Complex Numbers

1. Introduction

– Complex number is defined like this:

$$z = a + ib$$

where: z – complex number,
a,b – real numbers,
i – imaginary number with property $i^2=-1$,
a – real part,
ib – imaginary part.

– In polar coordinates complex number is defined like this:

$$z = |z| e^{i\varphi}$$

where: z – complex number,
|z| – amplitude of complex number z,
φ – angle of complex number z,
i – imaginary number wiyh property $i^2=-1$.

– Complex nmbers can be gaphicaly represented like this:

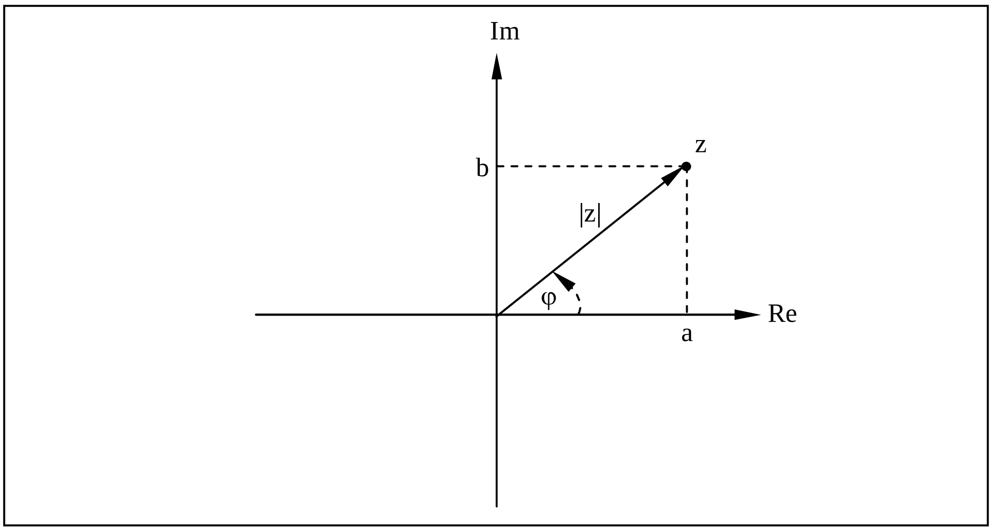

Figure 1.1. Graphical representation of complex numbers.

– In order to understand why complex numbers were invented and why they are defined as they are first you need to understand why imaginary number 'i' was invented which is explained in [1].

2. Adding complex numbers

- In a system of complex numbers '+' operation is defined like this:

$$z_1 = x_1+iy_1$$

$$z_2 = x_2+iy_2$$

$$z_3 = x_3+iy_3 \qquad (2.1)$$

$$z_3 = z_1+z_2$$

$$= (x_1+iy_1) + (x_2+iy_2)$$

$$= (x_1+ x_2) + i(y_1+y_2) \qquad (2.2)$$

- From (2.1) and (2.1) we finally get:

$$x_3 = x_1+ x_2$$

$$y_3 = y_1+y_2$$

that is:

$$x_3 + iy_3 = (x_1+ x_2) + i(y_1+y_2)$$

- This means that if we want do add two complex numbers we simply add their real parts and their imaginary parts separately
- If we represent each complex number with a vector, then '+' operations can be ilustrated as shown on following figure:

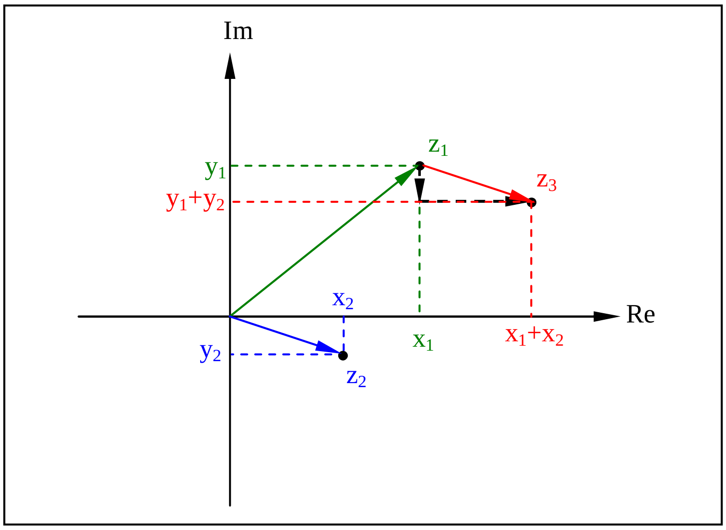

Figure 2.1. Adding two complex numbers by using vector's coordinates.

- Now that we have defined how this new '+' operation will work on complex numbers we need to check if this operation, when used on real numbers, will give the same results as the old '+' operation from the world of real numbers

2.1. Check back-compatibility

- Now we need to check if '+' operations, as defined above, gives the same result as we are used to when working only with real numbers.
- Since real numbers are subset of complex numbers when imaginary part is equal to zero, we can do our test like this:

$$z_1 = x_1+i0$$

$$z_2 = x_2+i0$$

$$\begin{aligned} z_3 &= x_3+iy_3 \\ &= (x_1+ x_2) + i(y_1+y_2) \\ &= (x_1+ x_2) + i(0+0) \\ &= x_1+ x_2 \end{aligned}$$

- Result is $x_3=x_1+x_2$ withouth imaginary part just as it should be.
- This means that our new '+' operation is defined in a way to be back-compatible with what we now of it in world of real numbers and this is why we can use it to define '+' operation for complex numbers.

3. Multipltying complex numbers

- In a system of complex numbers '·' operation is defined like this:

$$z_1 = A_1\angle\varphi_1$$

$$z_2 = A_2\angle\varphi_2$$

$$z_3 = z_1\cdot z_2$$

$$= (A_1\cdot A_2)\angle(\varphi_1+\varphi_2) \quad (2.3)$$

- From (2.3) and (2.4) we finally get:

$$A_3 = A_1\cdot A_2$$

$$\varphi_3 = \varphi_1+\varphi_2$$

that is:

$$A_3\angle\varphi_3 = (A_1\cdot A_2)\angle(\varphi_1+\varphi_2)$$

- This means that if we want do multiply two complex numbers we simply multiply lengths and add angles of their representing vectors as ilustrated on following figure:

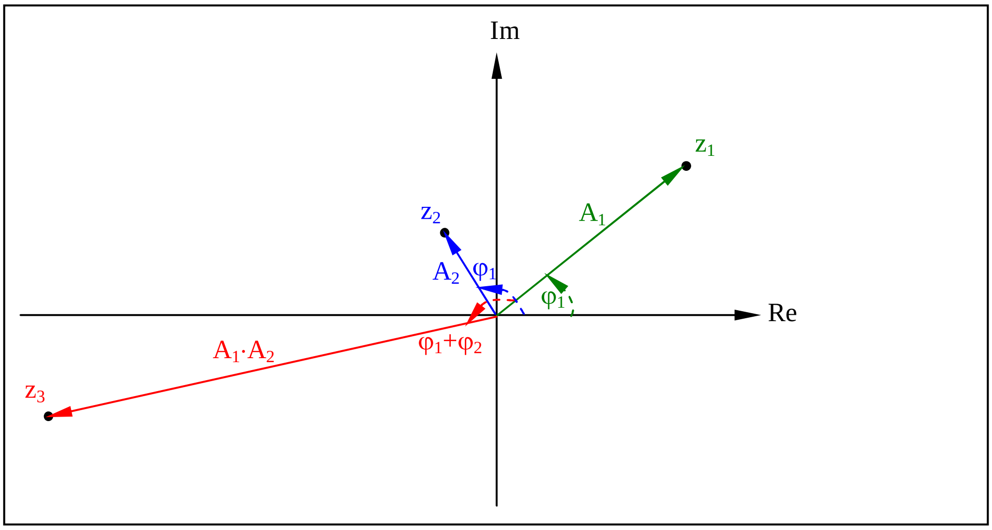

Figure 2.2. Multiplying two complex numbers by using vector's amplitude and angle.

- Now that we have defined how this new '·' operation will work on complex numbers we need to check if this operation:
  - ,when used on real numbers, will give the same results as the old '·' operation from the world of real numbers
  - would still give the result $i^2=-1$.

3.1. Check back-compatibility for two positive numbers

- Now we need to check if '·' operation, as defined above, gives the same result as we are used to when working only with real numbers.
- Since real numbers are subset of complex numbers when angle is zero for positive numbers and 180^o for negative numbers we can do our test like this:

$$z_1 = A_1\angle 0^o$$

$$z_2 = A_2\angle 0^o$$

$$z_3 = z_1 \cdot z_2$$

$$= (A_1 \cdot A_2)\angle(\varphi_1+\varphi_2)$$

$$= (A_1 \cdot A_2)\angle(0^o+0^o)$$

$$= (A_1 \cdot A_2)\angle 0^o$$

- We see that result of using newly defined '·' operation on two positive real numbers is again positive real number, like it should be in the world of real numbers, since the resulting complex number still has angle 0^o.

3.2. Check back-compatibility for two negative numbers

- Lets repeat the test for two negative numbers:

$$z_1 = A_1\angle 180^o$$

$$z_2 = A_2\angle 180^o$$

$$z_3 = z_1 \cdot z_2$$

$$= (A_1 \cdot A_2)\angle(\varphi_1+\varphi_2)$$

$$= (A_1 \cdot A_2)\angle(180^o+180^o)$$

$$= (A_1 \cdot A_2)\angle 360^o$$

$$= (A_1 \cdot A_2)\angle 0^o$$

- We see that result of using newly defined '·' operation on two negative real numbers is positive real number, like it should be in the world of real numbers, since the resulting complex number has angle 0^o.

3.3. Check back-compatibility for one negative and one positive number

- Lets repeat the test for one negative and one positive number:

$$z_1 = A_1\angle 180^o$$

$$z_2 = A_2\angle 0^o$$

$$z_3 = z_1 \cdot z_2$$

$$= (A_1 \cdot A_2)\angle(\varphi_1+\varphi_2)$$

$$= (A_1 \cdot A_2)\angle(180^o+0^o)$$

$$= (A_1 \cdot A_2)\angle 180^o$$

- We see that result of using newly defined '·' operation on one negative and one positive real number is negative real number, like it should be in the old world of real numbers, since the resulting complex number has angle 180^o.

3.4. Check back-compatibility for imaginary number

- Let us now test if newly defined '·' still gives following result:

$$i^2 = -1$$

- This relation can be interpreted like this:

$$i \cdot i = -1$$

so we can write:

$$z_1 = 1\angle 90^o$$

$$z_2 = 1\angle 90^o$$

$$z_3 = z_1 \cdot z_2$$

$$= (1 \cdot 1)\angle(90^o + 90^o)$$

$$= 1\angle 180^o$$

- Result is negative number, since angle is 180^o, of value 1 just as it should be.

- Introduction of imaginary number 'i' opened a door to the world of 2D numbers called complex numbers:

$$z = x + iy = A(\cos\varphi + i\sin\varphi) = Ae^{i\varphi}$$

4. References

[1] i - Imaginary Number.doc

5. Under Evaluation

http://mathews.ecs.fullerton.edu/c2000/