Complex logarithm function

1. Introduction

- Definition

– Complex logarithm is defined like this:

$$\log w = \log|w| + i(\angle w + k2\pi)$$

where: w – complex number,
|w| – amplitude of complex number w,
$\angle$w – angle of complex number w.

– Remainder of this chapter explaines why is complex algorithm defined as shown above.

- Defining problem

– Complex logarithm function is defined as inverse of complex exponential function [3]:

$$w = e^z \Big/ \log \tag{1.1}$$

$$\log w = \log e^z$$

$$= z \log e$$

$$= z \tag{1.2}$$

– Now if we define z like this:

$$z = x + iy \tag{1.3}$$

then in order to define complex logarithm logw we acctually need to define x and y.

- Preparing equations

– Inserting (1.3) into (1.1) we get:

$$w = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x \cos y + ie^x \sin y \tag{1.4}$$

– We will now introduce following variables:

$$a = e^x \cos y \tag{1.5}$$

$$b = e^x \sin y \tag{1.6}$$

– Inserting (1.5) and (1.6) into (1.4) we get:

$$w = a + ib \tag{1.7}$$

- Finding x

– To find x, we will square equations (1.5) and (1.6) and add them together:

$$a^2+b^2 = (e^x \cos y)^2 + (e^x \sin y)^2$$

$$= e^{2x} \cos^2 y + e^{2x} \sin^2 y$$

$$= e^{2x}\left(\cos^2 y + \sin^2 y\right)$$

$$= e^{2x} \quad \big/ \log$$

$$\log\left(a^2 + b^2\right) = \log e^{2x}$$

$$= 2x \log e$$

$$= 2x \quad \big/ \frac{1}{2}$$

$$x = \frac{1}{2}\log\left(a^2 + b^2\right)$$

$$= \log\left(a^2 + b^2\right)^{\frac{1}{2}}$$

$$= \log\sqrt{a^2 + b^2} \tag{1.8}$$

– Since w is defined with (1.7), equation (1.8) can be rewritten like this:

$$x = \log\left|w\right| \tag{1.9}$$

where: $|w|$ – magnitude of complex number w, the distance from w = a+ib to the origin in the complex plane.

- Finding y

– To solve for y we will divide equation (1.6) with (1.5) giving:

$$\frac{b}{a} = \frac{\sin y}{\cos y}$$

$$= \tan y$$

$$y = \arctan\frac{b}{a} \tag{1.10}$$

– Since w is defined with (1.7), equation (1.10) can be rewritten like this:

$$y = \angle w + k2\pi \tag{1.11}$$

- Final result

– And now to finally present our result using (1.2), (1.3), (1.9) and (1.11):

$$\log w = z$$

$$= x + iy$$

$$= \log\left|w\right| + i(\angle w + k2\pi)$$

2. References

[1] http://math.fullerton.edu/mathews/c2003/ComplexFunLogarithmMod.html

[2] Complex exponential function.doc

[3] http://mathforum.org/library/drmath/view/52230.html