Complex power of complex number

1. Introduction

– Comples power of complex number is defined like this;

$$x^{y} = |x|^{c}\, e^{-d\varphi}\left[\cos\left(c\varphi + d\ln|x|\right) + i\cos\left(c\varphi + d\ln|x|\right)\right] \tag{1.1}$$

where complex numbers x and y are defined like this:

$$x = |x|e^{i\varphi} \tag{1.2}$$

$$y = c + id \tag{1.3}$$

2. Proof of definition

– In this chapter we will prove relation (1.1):

$$x^{y} = |x|^{c}\, e^{-d\varphi}\left[\cos\left(c\varphi + d\ln|x|\right) + i\cos\left(c\varphi + d\ln|x|\right)\right] \tag{1.4}$$

– Using (1.2) and (1.3) we can write:

$$\begin{aligned} x^{y} &= \left(|x|e^{i\varphi}\right)^{c+id} \\ &= \left(|x|e^{i\varphi}\right)^{c}\left(|x|e^{i\varphi}\right)^{id} \\ &= |x|^{c} e^{ic\varphi}|x|^{id} e^{iid\varphi} \\ &= |x|^{c} e^{-d\varphi}e^{ic\varphi}|x|^{id} \end{aligned} \tag{1.5}$$

– To continue we need to prove following relation:

$$\begin{aligned} |x| &= e^{\ln|x|} \quad / \ln \\ \ln|x| &= \ln e^{\ln|x|} \\ &= \ln|x|\ln e \\ &= \ln|x| \end{aligned} \tag{1.6}$$

– Using (1.5), (1.4) can be written like:

$$\begin{aligned} x^{y} &= |x|^{c} e^{-d\varphi}e^{ic\varphi}\left(e^{\ln|x|}\right)^{id} \\ &= |x|^{c} e^{-d\varphi}e^{ic\varphi}e^{id\ln|x|} \\ &= |x|^{c} e^{-d\varphi}\left[\cos(c\varphi) + i\sin(c\varphi)\right]\left[\cos\left(d\ln|x|\right) + i\sin\left(d\ln|x|\right)\right] \end{aligned}$$

$$x^{y} = |x|^{c}\, e^{-d\varphi}\left\{\cos(c\varphi)\cos\left(d\ln|x|\right) - \sin(c\varphi)\sin\left(d\ln|x|\right) + i\left[\sin(c\varphi)\cos\left(d\ln|x|\right) + \cos(c\varphi)\sin\left(d\ln|x|\right)\right]\right\} \tag{1.7}$$

– Using trigonometric functions [2]:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{1.8}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{1.9}$$

(1.6) can be rewritten like this:

$$x^{y} = |x|^{c}\, e^{-d\varphi}\left[\cos\left(c\varphi + d\ln|x|\right) + j\cos\left(c\varphi + d\ln|x|\right)\right] \tag{1.10}$$

3. Proof of i^i

– We will now prove that i^i is real number starting with Euler's formula [1]:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{2.1}$$

– Choosing:

$$\varphi = \frac{\pi}{2}$$

formula (2.1) becomes:

$$\begin{aligned} e^{i\frac{\pi}{2}} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i\cdot 1 \\ &= i \end{aligned} \tag{2.2}$$

– Using (2.2) we can write:

$$\begin{aligned} i^i &= \left(e^{i\frac{\pi}{2}}\right)^i \\ &= e^{i^2\frac{\pi}{2}} \\ &= e^{-\frac{\pi}{2}} \\ &= 0.20787957635 \end{aligned}$$

4. References

[1] http://home.att.net/~srschmitt/complexnumbers.html

[2] Trigonometric Functions.doc