Derivations

1. Properties of derivation

- In this chapter we shall present properties of derivation.
- Product rule: $$\frac{d}{dx}\left[f(x)g(x)\right] = f\frac{dg}{dx} + g\frac{df}{dx} \tag{1.1}$$
- Chain rule: $$\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx} \tag{1.2}$$

2. Product rule

– Product rule is defined like this [1]:

$$(fg)' = f'g + fg' \tag{2.1}$$

or in Leibnitz notation like this:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{2.2}$$

2.1. Proof of product rule

– In this chapter we shall prove product rule (1.2) [1]:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{2.3}$$

– We shall start by using chain rules:

$$(u+v)^2 = u^2 + 2uv + v^2 \tag{2.4}$$

$$(u-v)^2 = u^2 - 2uv + v^2 \tag{2.5}$$

to prove:

$$\begin{aligned} uv &= \frac{1}{4}\left[(u+v)^2 - (u-v)^2\right] \qquad (2.6) \\ &= \frac{1}{4}\left[u^2 + 2uv + v^2 - u^2 + 2uv - v^2\right] \\ &= \frac{1}{4}\left[2uv + 2uv\right] \\ &= \frac{1}{4}4uv \\ &= uv \end{aligned}$$

– We shall now prove (1.3) by taking derivative of (1.6):

$$\begin{aligned} \frac{d}{dx}(uv) &= \frac{d}{dx}\left\{\frac{1}{4}\left[(u+v)^2 - (u-v)^2\right]\right\} \\ &= \frac{1}{4}\left[2(u+v)\left(\frac{du}{dx} + \frac{dv}{dx}\right) - 2(u-v)\left(\frac{du}{dx} - \frac{dv}{dx}\right)\right] \\ &= \frac{1}{4}\left[2\left(\cancel{u\frac{du}{dx}} + v\frac{du}{dx} + u\frac{dv}{dx} + \cancel{v\frac{dv}{dx}}\right) - 2\left(\cancel{u\frac{du}{dx}} - v\frac{du}{dx} - u\frac{dv}{dx} + \cancel{v\frac{dv}{dx}}\right)\right] \\ &= \frac{1}{4}\left[2v\frac{du}{dx} + 2u\frac{dv}{dx} + 2v\frac{du}{dx} + 2u\frac{dv}{dx}\right] \\ &= \frac{1}{4}\left[4v\frac{du}{dx} + 4u\frac{dv}{dx}\right] \\ &= u\frac{dv}{dx} + v\frac{du}{dx} \end{aligned}$$

3. Chain rule

– Chain rule is formula for derivative of the composite of two functions:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{df}{dg}\frac{dg}{dx} \tag{2.7}$$

3.1. Proof of chain rule

– In this chapter we shall prove chain rule (1.7) [3]:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{df}{dg}\frac{dg}{dx}$$

– We start by using definition of derivative for g(x):

$$\frac{d}{dx} g(x) = \frac{g(x+dx)-g(x)}{dx} \quad \Big/ \; dx$$

$$\frac{dg}{dx} dx = g(x+dx)-g(x)$$

$$g(x+dx) = g(x)+\frac{dg}{dx} dx \tag{2.8}$$

– For f(g) function (1.8) can be written like this:

$$f(g+dg) = f(g)+\frac{df}{dg} dg \tag{2.9}$$

– Using definition of derivative f(g(x)) can be written like this:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{f\big(g(x+dx)\big)-f\big(g(x)\big)}{dx} \tag{2.10}$$

– Inserting (1.8) into numerator we get:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{f\left(g(x)+\frac{dg}{dx} dx\right)-f\big(g(x)\big)}{dx} \tag{2.11}$$

– Using (1.9), left part of the numerator can be rewritten like:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{\cancel{f\big(g(x)\big)}+\frac{df}{dg}\left(\frac{dg}{dx} dx\right)-\cancel{f\big(g(x)\big)}}{dx}$$

$$= \frac{\frac{df}{dg}\frac{dg}{dx} dx}{dx}$$

$$= \frac{df}{dg}\frac{dg}{dx} \tag{2.12}$$

4. References

[1] http://en.wikipedia.org/wiki/Product_rule

[2] http://en.wikipedia.org/wiki/Chain_rule

[3] http://people.hofstra.edu/Stefan_Waner/RealWorld/proofs/chainruleproof.html