DURBIN ALGORITHM

1.1. Durbin's algorithm

- Durbin algorithm is convinient way of solwing following matrix equation where left most matrix is Toeplitz matrix:

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_{p-1} \\ r_1 & r_0 & \cdots & r_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p-1} & r_{p-2} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix} \tag{1.1}$$

where: r_i – known values.
a_i – unknown values that we want to find.

– Following example of (1.1) demonstrates that in Toeplitz matrix each descending diagonal from left to right is constant:

$$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_1 & r_2 \\ r_2 & r_1 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{1.2}$$

– We will create steps of Durbin algorithm by solving few matrix equations to try to see any emerging rules.
– Extra info about durbin algorithm can be found in reference [1] pages 106,115, & 117.

- Durbin algorithm is defined with the following set of equations which should be calculated in the order af appearence:

$$E_0 = r_0 \tag{1.3}$$

$$a_{i/i} = \frac{r_i - \sum_{j=1}^{i-1} a_{j/(i-1)} r_{i-j}}{E_{i-1}} \;;\; 1 \le i \le p \tag{1.4}$$

$$a_{j/i} = a_{j/(i-1)} - a_{i/i} a_{(i-j)/(i-1)} \;;\; 1 \le j < i \tag{1.5}$$

$$E_i = (1 - a_{i/i}^2) E_{i-1} \tag{1.6}$$

where: E_i – helper variable which makes equation (1.4) simpler.

– What Durbin algorithm does is that it first solves matrix equation for p=1 calculating $a_{1/1}$.
– Then while calculating matrix equation for p=2 it uses $a_{1/1}$ which was calculated in previous step.
– Then while calculating matrix equation for p=3 it uses all fo the results obtained from previous steps.
– This continues for each higher value of p.

– When Durbin algorithm is used to calculate LPC - Linear Predictive Coefficients, above variables have following meaning:

r_i – autocorelation coefficients calculated from the signal,
$a_{1/j}$ – Linear Predictive Coefficients (LPC) we are looking for,
p – number of LPCs.

– For iteration i=5 equations (1.4) to (1.6) become:

$$a_{5/5} = \frac{r_5 - a_{1/4} r_4 - a_{2/4} r_3 - a_{3/4} r_2 - a_{4/4} r_1}{(1 - a_{4/4}^2)(1 - a_{3/3}^2)(1 - a_{2/2}^2)(1 - a_{1/1}^2) r_0}$$

$$a_{1/5} = a_{1/4} - a_{5/5} a_{4/4}$$

$$a_{2/5} = a_{2/4} - a_{5/5} a_{3/4}$$

$$a_{3/5} = a_{3/4} - a_{5/5} a_{2/4}$$

$$a_{4/5} = a_{4/4} - a_{5/5} a_{1/4}$$

$$E_5 = (1 - a_{5/5}^2) E_4$$

- Equations (1.4) and (1.5) might be easier to understand and remember if we present them in the matrix form:

$$a_{i/i} = \frac{r_i - \begin{bmatrix} r_{i-1} & \cdots & r_2 & r_1 \end{bmatrix} \begin{bmatrix} a_{1/i-1} \\ a_{2/i-1} \\ \vdots \\ a_{i-1/i-1} \end{bmatrix}}{E_{i-1}} \tag{1.7}$$

$$\begin{bmatrix} a_{1/i} \\ a_{2/i} \\ \vdots \\ a_{i-1/i} \end{bmatrix} = \begin{bmatrix} a_{1/i-1} \\ a_{2/i-1} \\ \vdots \\ a_{i-1/i-1} \end{bmatrix} - a_{i/i} \begin{bmatrix} a_{i-1/i-1} \\ \vdots \\ a_{2/i-1} \\ a_{1/i-1} \end{bmatrix} \tag{1.8}$$

– Using equations (1.7) and (1.8) we can now rewrite equations for Durbin algorithm in matrix form as follows:

$$E_0 = r_0 \tag{1.9}$$

$$a_{i/i} = \frac{r_i - R_{i-1} A_{i-1/i-1}}{E_{i-1}} \tag{1.10}$$

$$A_{i-1/i} = A_{i-1/i-1} - a_{i/i} A^{flip}_{i-1/i-1} \qquad 1 \le i \le p \tag{1.11}$$

$$E_i = (1 - a_{i/i}^2) E_{i-1} \tag{1.12}$$

where: $R_i = [r_i \quad r_{i-1} \quad \cdots \quad r_1]$

$$A_{i/j} = \begin{bmatrix} a_{1/j} \\ a_{2/j} \\ \vdots \\ a_{i/j} \end{bmatrix}$$

$$A^{flip}_{i/j} = \begin{bmatrix} a_{i/j} \\ a_{i-1/j} \\ \vdots \\ a_{1/j} \end{bmatrix}$$

– Examples of formulas (1.7) and (1.8), that is (1.10) and(1.11) for iteration i=5 would look like this:

$$a_{5/5} = \frac{r_5 - \begin{bmatrix} r_4 & r_3 & r_2 & r_1 \end{bmatrix} \begin{bmatrix} a_{1/4} \\ a_{2/4} \\ a_{3/4} \\ a_{4/4} \end{bmatrix}}{E_4} = \frac{r_5 - r_4 a_{1/4} - r_3 a_{2/4} - r_2 a_{3/4} - r_1 a_{4/4}}{(1-a_{44}^2)(1-a_{33}^2)(1-a_{22}^2)(1-a_{11}^2) r_0}$$

$$\begin{bmatrix} a_{1/5} \\ a_{2/5} \\ a_{3/5} \\ a_{4/5} \end{bmatrix} = \begin{bmatrix} a_{1/4} \\ a_{2/4} \\ a_{3/4} \\ a_{4/4} \end{bmatrix} - a_{5/5} \begin{bmatrix} a_{4/4} \\ a_{3/4} \\ a_{2/4} \\ a_{1/4} \end{bmatrix}$$

- We will show later that equation (1.7), that is (1.10), can be written in following way:

$$a_{i/i}=\frac{r_i-\begin{bmatrix}r_{i-1} & \cdots & r_2 & r_1\end{bmatrix}\begin{bmatrix}a_{1/i-1}\\ a_{2/i-1}\\ \vdots\\ a_{i-1/i-1}\end{bmatrix}}{r_0-\begin{bmatrix}r_1 & r_1 & \cdots & r_{i-1}\end{bmatrix}\begin{bmatrix}a_{1/i-1}\\ a_{2/i-1}\\ \vdots\\ a_{i-1/i-1}\end{bmatrix}} \qquad (1.13)$$

For iteration i=5, (1.13) would look like this:

$$a_{5/5}=\frac{r_5-\begin{bmatrix}r_4 & r_3 & r_2 & r_1\end{bmatrix}\begin{bmatrix}a_{1/4}\\ a_{2/4}\\ a_{3/4}\\ a_{4/4}\end{bmatrix}}{r_0-\begin{bmatrix}r_1 & r_2 & r_3 & r_4\end{bmatrix}\begin{bmatrix}a_{1/4}\\ a_{2/4}\\ a_{3/4}\\ a_{4/4}\end{bmatrix}}=\frac{r_5-r_4a_{1/4}-r_3a_{2/4}-r_2a_{3/4}-r_1a_{4/4}}{r_0-r_1a_{1/4}-r_2a_{2/4}-r_3a_{3/4}-r_4a_{4/4}}$$

– This property is used later to show that result of solving (1.1) using matrix algebra is the same as the one obtained using (1.10).

- In later chapters we would also need variable value which is by Durban algorithm calculated as folows:
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$$G^2=E_0E_p \qquad (1.14)$$

Equation (1.14) can also be written as:

$$G^2=1-\frac{1}{r_0}\sum_{k=1}^{p}a_kr_k$$

or in matrix form as:

$$G^2=1-\frac{1}{r_0}\begin{bmatrix}a_1 & a_2 & \cdots & a_p\end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ \vdots\\ r_p\end{bmatrix} \qquad (1.15)$$

- In following 3 chapters we will show how Durbin algorithm was created.

– We will do this by solving matrix equation (1.1) when p is 1,2 and 3 using simple matrix algebra.
– In obtained results we will find a pattern that can be used to find solution for matrix equation (1.1) for any p.
– In each chapter we compare our results with those of Durbin algorithm to show that pattern we have found really is what is now called Durbin algorithm.

1.1.1. Matrix equation when p=1

Matrix equation (1.1) for first order polinom, when p=1, looks like this:

$$[r_0][a_{1/1}] = [r_1] \tag{1.16}$$

1.1.1.1. Solving matrix equation when p=1 using Durbin algorithm

Solving (1.4) for i = 1 we get:

$$a_{1/1} = \frac{r_1 - \sum_{j=1}^{1-1} a_{j/(1-1)} r_{1-j}}{E_{1-1}} \tag{1.17}$$

– Since the upper sumation limit in (1.17) is lower then starting value of j, the sum is zero.
– Together with inserting (1.3) into (1.17) we get:

$$\boxed{a_{1/1} = \frac{r_1}{r_0}} \tag{1.18}$$

1.1.1.2. Solving matrix equation when p=1 using simple matrix algebra

From (1.16) we get:

$$r_0 a_{1/1} = r_1 \tag{1.19}$$

which leads to:

$$\boxed{a_{1/1} = \frac{r_1}{r_0}} \tag{1.20}$$

Since (1.20) and (1.18) are the same that means that Durbin algorithm is valid for p = 1;

1.1.2. Matrix equation when p=2

Matrix equation (1.1) for second order polinom looks like this:

$$\begin{bmatrix} r_0 & r_1 \\ r_1 & r_0 \end{bmatrix} \begin{bmatrix} a_{1/2} \\ a_{2/2} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{1.21}$$

1.1.2.1. Solving matrix equation when p=2 using Durbin algorithm

Solving (1.6) for i = 1 we get:

$$E_1 = (1 - a_{1/1}^2) E_{1-1} \tag{1.22}$$

Inserting (1.3) into (1.22) we get:

$$E_1 = (1 - a_{1/1}^2) r_0 \tag{1.23}$$

Solving (1.4) for i = 2 we get:

$$a_{2/2} = \frac{r_2 - \sum_{j=1}^{2-1} a_{j/(2-1)} r_{2-j}}{E_{2-1}} = \frac{r_2 - \sum_{j=1}^{1} a_{j/1} r_{2-j}}{E_1} \tag{1.24}$$

$$a_{2/2} = \frac{r_2 - a_{1/1} r_1}{E_1} \tag{1.25}$$

Inserting (1.23) into (1.25) we get:

$$\boxed{a_{2/2} = \frac{r_2 - a_{1/1} r_1}{(1 - a_{1/1}^2) r_0}} \tag{1.26}$$

Solving (1.5) for i = 2 and j = 1 we get:

$$a_{1/2} = a_{1/(2-1)} - a_{2/2} a_{(2-1)/(2-1)}$$

$$\boxed{a_{1/2} = a_{1/1} - a_{2/2} a_{1/1}} \tag{1.27}$$

Result of Durbin algorithm for p=2 are equations (1.26) and (1.27).

1.1.2.2. Solving matrix equation when p=2 using simple matrix algebra

- From (1.21) we get folowing two equations:

$$r_0 a_{1/2} + r_1 a_{2/2} = r_1 \tag{1.28}$$

$$r_1 a_{1/2} + r_0 a_{2/2} = r_2 \tag{1.29}$$

- From (1.28) we get:

$$r_0 a_{1/2} + r_1 a_{2/2} = r_1 \quad \Big/ \frac{1}{r_0}$$

$$a_{1/2} + \frac{r_1}{r_0} a_{2/2} = \frac{r_1}{r_0}$$

$$a_{1/2} = \frac{r_1}{r_0} - \frac{r_1}{r_0} a_{2/2} \tag{1.30}$$

Introducing (1.21) into (1.30) we get:

$$\boxed{a_{1/2} = a_{1/1} - a_{1/1} a_{2/2}} \tag{1.31}$$

- Since (1.31) is the same as (1.27), that means that Durbin algorithm is valid for $a_{1/2}$.

From (1.29) we get:

$$r_1 a_{1/2} + r_0 a_{2/2} = r_2 \quad \Big/ \frac{1}{r_0}$$

$$\frac{r_1}{r_0}a_{1/2}+a_{2/2}=\frac{r_2}{r_0}$$

$$a_{2/2}=\frac{r_2}{r_0}-\frac{r_1}{r_0}a_{1/2} \tag{1.32}$$

Introducing (1.20) into (1.32) we get:

$$a_{2/2}=\frac{r_2}{r_0}-a_{1/1}a_{1/2} \tag{1.33}$$

- Introducing (1.31) into (1.33) we get:

$$a_{2/2}=\frac{r_2}{r_0}-a_{1/1}(a_{1/1}-a_{1/1}a_{2/2})$$

$$a_{2/2}=\frac{r_2}{r_0}-a_{1/1}^2+a_{1/1}^2a_{2/2}$$

$$a_{2/2}(1-a_{1/1}^2)=\frac{r_2}{r_0}-a_{1/1}^2$$

$$a_{2/2}=\frac{\frac{r_2}{r_0}-a_{1/1}^2}{1-a_{1/1}^2}$$

$$a_{2/2}=\frac{r_2-a_{1/1}^2r_0}{\left(1-a_{1/1}^2\right)r_0}$$

Introducing (1.20) we get:

$$a_{2/2}=\frac{r_2-\left(\frac{r_1}{r_0}\right)^2r_0}{\left(1-a_{1/1}^2\right)r_0}$$

$$a_{2/2}=\frac{r_2-\frac{r_1}{r_0}r_1}{\left(1-a_{1/1}^2\right)r_0}$$

Using (1.20) once more we get:

$$\boxed{a_{2/2}=\frac{r_2-a_{1/1}r_1}{\left(1-a_{1/1}^2\right)r_0}} \tag{1.34}$$

Since (1.34) is equal to (1.26) that means that Durbin algorithm is valid for $a_{2/2}$.

- We shall end this chapter by finding alternative way of writing (1.31)

Introducing (1.34) into (1.31) we get:

$$a_{1/2}=a_{1/1}-a_{1/1}\frac{r_2-a_{1/1}r_1}{\left(1-a_{1/1}^2\right)r_0}$$

$$a_{1/2}=\frac{a_{1/1}\left(1-a_{1/1}^2\right)r_0-a_{1/1}\left(r_2-a_{1/1}r_1\right)}{\left(1-a_{1/1}^2\right)r_0}$$

$$a_{1/2}=\frac{a_{1/1}r_0-a_{1/1}^3r_0-a_{1/1}r_2+a_{1/1}^2r_1}{\left(1-a_{1/1}^2\right)r_0}$$

$$a_{1/2}=\frac{\frac{r_1}{r_0}r_0-\frac{r_1^3}{r_0^3}r_0-a_{1/1}r_2+\frac{r_1^2}{r_0^2}r_1}{\left(1-a_{1/1}^2\right)r_0}$$

$$a_{1/2}=\frac{r_1-a_{1/1}r_2-\frac{r_1^3}{r_0^2}+\frac{r_1^3}{r_0^2}r_1}{\left(1-a_{1/1}^2\right)r_0}$$

$$\boxed{a_{1/2} = \frac{r_1 - a_{1/1} r_2}{\left(1 - a_{1/1}^2\right) r_0}} \tag{1.35}$$

1.1.3. Matrix equation when p=3

Matrix equation (1.1) for third order polinom looks like this:

$$\begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix} \begin{bmatrix} a_{1/3} \\ a_{2/3} \\ a_{3/3} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{1.36}$$

1.1.3.1. Solving matrix equation when p=3 using Durbin algorithm

Solving (1.6) for i = 2 we get:

$$E_2 = (1 - a_{2/2}^2) E_{2-1} \tag{1.37}$$

$$E_2 = (1 - a_{2/2}^2) E_1 \tag{1.38}$$

Inserting (1.23) into (1.38) we get:

$$E_2 = (1 - a_{2/2}^2)(1 - a_{1/1}^2) r_0 \tag{1.39}$$

Solving (1.4) for i = 3 we get:

$$a_{3/3} = \frac{r_3 - \sum_{j=1}^{3-1} a_{j/(3-1)} r_{3-j}}{E_{3-1}} = \frac{r_3 - \sum_{j=1}^{2} a_{j/2} r_{3-j}}{E_2} = \frac{r_3 - a_{1/2} r_{3-1} - a_{2/2} r_{3-2}}{E_2} \tag{1.40}$$

$$a_{3/3} = \frac{r_3 - a_{1/2} r_2 - a_{2/2} r_1}{E_2} \tag{1.41}$$

Inserting (1.39) into (1.41) we get:

$$\boxed{a_{3/3} = \frac{r_3 - a_{1/2} r_2 - a_{2/2} r_1}{(1 - a_{2/2}^2)(1 - a_{1/1}^2) r_0}} \tag{1.42}$$

Solving (1.5) for i = 3 and j = 1 we get:

$$a_{1/3} = a_{1/(3-1)} - a_{3/3} a_{(3-1)/(3-1)} \tag{1.43}$$

$$\boxed{a_{1/3} = a_{1/2} - a_{3/3} a_{2/2}} \tag{1.44}$$

Solving (1.5) for i = 3 and j = 2 we get:

$$a_{2/3} = a_{2/(3-1)} - a_{3/3} a_{(3-2)/(3-1)}$$

$$\boxed{a_{2/3} = a_{2/2} - a_{3/3} a_{1/2}} \tag{1.45}$$

Result of Durbin algorithm for p=3 are equations **Error! Not a valid link.**, **Error! Not a valid link.** and **Error! Not a valid link.**.

1.1.3.2. Solving matrix equation when p=3 using simple matrix algebra

- From (1.36) we get folowing three equations:

$$r_0 a_{1/3} + r_1 a_{2/3} + r_2 a_{3/3} = r_1 \tag{1.46}$$

$$r_1 a_{1/3} + r_0 a_{2/3} + r_1 a_{3/3} = r_2 \tag{1.47}$$

$$r_2 a_{1/3} + r_1 a_{2/3} + r_0 a_{3/3} = r_3 \tag{1.48}$$

- From (1.46) we get:

$$r_0 a_{1/3} + r_1 a_{2/3} + r_2 a_{3/3} = r_1 \quad \Big/ \frac{1}{r_0}$$

$$a_{1/3} + \frac{r_1}{r_0} a_{2/3} + \frac{r_2}{r_0} a_{3/3} = \frac{r_1}{r_0}$$

$$a_{1/3} = \frac{r_1}{r_0} - \frac{r_1}{r_0} a_{2/3} - \frac{r_2}{r_0} a_{3/3} \tag{1.49}$$

From (1.47) we get:

$$r_1 a_{1/3} + r_0 a_{2/3} + r_1 a_{3/3} = r_2 \;\Big/\; \frac{1}{r_0}$$

$$\frac{r_1}{r_0} a_{1/3} + a_{2/3} + \frac{r_1}{r_0} a_{3/3} = \frac{r_2}{r_0}$$

$$a_{2/3} = \frac{r_2}{r_0} - \frac{r_1}{r_0} a_{1/3} - \frac{r_1}{r_0} a_{3/3} \tag{1.50}$$

Introducing (1.49) into (1.50) we get:

$$a_{2/3} = \frac{r_2}{r_0} - \frac{r_1}{r_0}\left(\frac{r_1}{r_0} - \frac{r_1}{r_0} a_{2/3} - \frac{r_2}{r_0} a_{3/3}\right) - \frac{r_1}{r_0} a_{3/3}$$

$$a_{2/3} = \frac{r_2}{r_0} - \left(\frac{r_1}{r_0}\right)^2 + \left(\frac{r_1}{r_0}\right)^2 a_{2/3} + \frac{r_1}{r_0}\frac{r_2}{r_0} a_{3/3} - \frac{r_1}{r_0} a_{3/3}$$

$$a_{2/3}\left[1 - \left(\frac{r_1}{r_0}\right)^2\right] = \frac{r_2}{r_0} - \left(\frac{r_1}{r_0}\right)^2 + a_{3/3}\left(\frac{r_1}{r_0}\frac{r_2}{r_0} - \frac{r_1}{r_0}\right)$$

$$a_{2/3} = \frac{\frac{r_2}{r_0} - \left(\frac{r_1}{r_0}\right)^2 + a_{3/3}\left(\frac{r_1}{r_0}\frac{r_2}{r_0} - \frac{r_1}{r_0}\right)}{1 - \left(\frac{r_1}{r_0}\right)^2}\frac{r_0}{r_0}$$

$$a_{2/3} = \frac{r_2 - \frac{r_1}{r_0} r_1 + a_{3/3}\left(\frac{r_1}{r_0} r_2 - r_1\right)}{\left[1 - \left(\frac{r_1}{r_0}\right)^2\right] r_0}$$

Introducing (1.20) we get:

$$a_{2/3} = \frac{r_2 - a_{1/1} r_1 + a_{3/3}\left(a_{1/1} r_2 - r_1\right)}{\left(1 - a_{1/1}^2\right) r_0}$$

$$a_{2/3} = \frac{r_2 - a_{1/1} r_1 - a_{3/3}\left(r_1 - a_{1/1} r_2\right)}{\left(1 - a_{1/1}^2\right) r_0}$$

$$a_{2/3} = \frac{r_2 - a_{1/1} r_1}{\left(1 - a_{1/1}^2\right) r_0} - a_{3/3}\frac{\left(r_1 - a_{1/1} r_2\right)}{\left(1 - a_{1/1}^2\right) r_0}$$

Introducing (1.34) we get:

$$a_{2/3} = a_{2/2} - a_{3/3}\frac{\left(r_1 - a_{1/1} r_2\right)}{\left(1 - a_{1/1}^2\right) r_0} \tag{1.51}$$

Introducing (1.35) into (1.51) we get:

$$\boxed{a_{2/3} = a_{2/2} - a_{3/3} a_{1/2}} \tag{1.52}$$

Since (1.52) is the same as (1.45), that means that Durbin algorithm is valid for $a_{2/3}$.

- Introducing (1.51) into (1.49) we get:

$$a_{1/3} = \frac{r_1}{r_0} - \frac{r_1}{r_0}\left(a_{2/2} - a_{3/3}\frac{\left(r_1 - a_{1/1} r_2\right)}{\left(1 - a_{1/1}^2\right) r_0}\right) - \frac{r_2}{r_0} a_{3/3}$$

$$a_{1/3} = \frac{r_1}{r_0} - \frac{r_1}{r_0} a_{2/2} + \frac{r_1}{r_0} a_{3/3}\frac{\left(r_1 - a_{1/1} r_2\right)}{\left(1 - a_{1/1}^2\right) r_0} - \frac{r_2}{r_0} a_{3/3}$$

$$a_{1/3}=\frac{r_1}{r_0}-\frac{r_1}{r_0}a_{2/2}+a_{3/3}\left(\frac{r_1}{r_0}\frac{(r_1-a_{1/1}r_2)}{(1-a_{1/1}^2)r_0}-\frac{r_2}{r_0}\right)$$

Introducing (1.20) we get:

$$a_{1/3}=a_{1/1}-a_{1/1}a_{2/2}+a_{3/3}\left(a_{1/1}\frac{(r_1-a_{1/1}r_2)}{(1-a_{1/1}^2)r_0}-\frac{r_2}{r_0}\right)$$

$$a_{1/3}=a_{1/1}-a_{1/1}a_{2/2}-a_{3/3}\left(\frac{r_2}{r_0}-a_{1/1}\frac{(r_1-a_{1/1}r_2)}{(1-a_{1/1}^2)r_0}\right)$$

Introducing (1.31) we get:

$$a_{1/3}=a_{1/2}-a_{3/3}\left(\frac{r_2}{r_0}-a_{1/1}\frac{(r_1-a_{1/1}r_2)}{(1-a_{1/1}^2)r_0}\right)$$

$$a_{1/3}=a_{1/2}-a_{3/3}\frac{(1-a_{1/1}^2)r_2-a_{1/1}(r_1-a_{1/1}r_2)}{(1-a_{1/1}^2)r_0}$$

$$a_{1/3}=a_{1/2}-a_{3/3}\frac{r_2-a_{1/1}^2r_2-a_{1/1}r_1+a_{1/1}^2r_2}{(1-a_{1/1}^2)r_0}$$

$$a_{1/3}=a_{1/2}-a_{3/3}\frac{r_2-a_{1/1}r_1}{(1-a_{1/1}^2)r_0}$$

Introducing (1.34) we get:

$$\boxed{a_{1/3}=a_{1/2}-a_{3/3}a_{2/2}} \qquad (1.53)$$

Since (1.44) is equal to (1.53) that means that Durbin algorith is valid for $a_{1/3}$.

- From (1.48) we get:

$$r_2a_{1/3}+r_1a_{2/3}+r_0a_{3/3}=r_3 \Big/ \frac{1}{r_0} \qquad (1.54)$$

$$\frac{r_2}{r_0}a_{1/3}+\frac{r_1}{r_0}a_{2/3}+a_{3/3}=\frac{r_3}{r_0}$$

$$a_{3/3}=\frac{r_3}{r_0}-\frac{r_2}{r_0}a_{1/3}-\frac{r_1}{r_0}a_{2/3} \qquad (1.55)$$

Introducing (1.52) and (1.53) into (1.55) we get:

$$a_{3/3}=\frac{r_3}{r_0}-\frac{r_2}{r_0}(a_{1/2}-a_{3/3}a_{2/2})-\frac{r_1}{r_0}(a_{2/2}-a_{3/3}a_{1/2})$$

$$a_{3/3}=\frac{r_3}{r_0}-\frac{r_2}{r_0}a_{1/2}+\frac{r_2}{r_0}a_{3/3}a_{2/2}-\frac{r_1}{r_0}a_{2/2}+\frac{r_1}{r_0}a_{3/3}a_{1/2}$$

$$a_{3/3}\left(1-\frac{r_2}{r_0}a_{2/2}-\frac{r_1}{r_0}a_{1/2}\right)=\frac{r_3}{r_0}-\frac{r_2}{r_0}a_{1/2}-\frac{r_1}{r_0}a_{2/2}$$

$$a_{3/3}=\frac{\frac{r_3}{r_0}-\frac{r_2}{r_0}a_{1/2}-\frac{r_1}{r_0}a_{2/2}}{1-\frac{r_2}{r_0}a_{2/2}-\frac{r_1}{r_0}a_{1/2}}\frac{r_0}{r_0}$$

$$a_{3/3}=\frac{r_3-r_2a_{1/2}-r_1a_{2/2}}{r_0-r_2a_{2/2}-r_1a_{1/2}}$$

$$\boxed{a_{3/3}=\frac{r_3-r_2a_{1/2}-r_1a_{2/2}}{r_0-r_1a_{1/2}-r_2a_{2/2}}} \qquad (1.56)$$

Equation (1.56) has the same numerator as (1.42), so in order to prove that this two equations are equal we have to prove that they have equal denumerators.

$$r_0 - r_2 a_{2/2} - r_1 a_{1/2} = (1 - a_{2/2}^2)(1 - a_{1/1}^2) r_0 \tag{1.57}$$

By inserting (1.35) instead $a_{1/2}$ and (1.34) instead $a_{2/2}$ into (1.57) we get:

$$r_0 - r_2 \frac{r_2 - a_{1/1} r_1}{(1 - a_{1/1}^2) r_0} - r_1 \frac{r_1 - a_{1/1} r_2}{(1 - a_{1/1}^2) r_0} = (1 - \left(\frac{r_2 - a_{1/1} r_1}{(1 - a_{1/1}^2) r_0} \right)^2)(1 - a_{1/1}^2) r_0$$

$$\frac{(1 - a_{1/1}^2) r_0^2 - r_2^2 + r_2 a_{1/1} r_1 - r_1^2 + r_1 a_{1/1} r_2}{(1 - a_{1/1}^2) r_0} = \frac{(1 - a_{1/1}^2)^2 r_0^2 - (r_2 - a_{1/1} r_1)^2}{(1 - a_{1/1}^2)^2 r_0^2} (1 - a_{1/1}^2) r_0$$

$$(1 - a_{1/1}^2) r_0^2 - r_2^2 + r_2 a_{1/1} r_1 - r_1^2 + r_1 a_{1/1} r_2 = (1 - a_{1/1}^2)^2 r_0^2 - (r_2 - a_{1/1} r_1)^2$$

$$r_0^2 - a_{1/1}^2 r_0^2 - r_1^2 - r_2^2 + 2 r_1 r_2 a_{1/1} = (1 + a_{1/1}^4 - 2 a_{1/1}^2) r_0^2 - r_2^2 - a_{1/1}^2 r_1^2 + 2 r_1 r_2 a_{1/1}$$

$$r_0^2 - a_{1/1}^2 r_0^2 - r_1^2 = r_0^2 + a_{1/1}^4 r_0^2 - 2 a_{1/1}^2 r_0^2 - a_{1/1}^2 r_1^2$$

$$a_{1/1}^2 r_0^2 - r_1^2 = a_{1/1}^4 r_0^2 - a_{1/1}^2 r_1^2$$

$$\frac{r_1^2}{r_0^2} r_0^2 - r_1^2 = \frac{r_1^4}{r_0^4} r_0^2 - \frac{r_1^2}{r_0^2} r_1^2$$

$$r_1^2 - r_1^2 = \frac{r_1^4}{r_0^2} - \frac{r_1^4}{r_0^2}$$

$$0 = 0$$

1.2. References

[1] Fundamentals of speech recognition", Lawrence Rabiner & Biing-Hwang Juang