1. Exponential function

– Exponential function is defined like:

$$f(x) = e^x \tag{1.1}$$

and it's graph is shown on following figure:

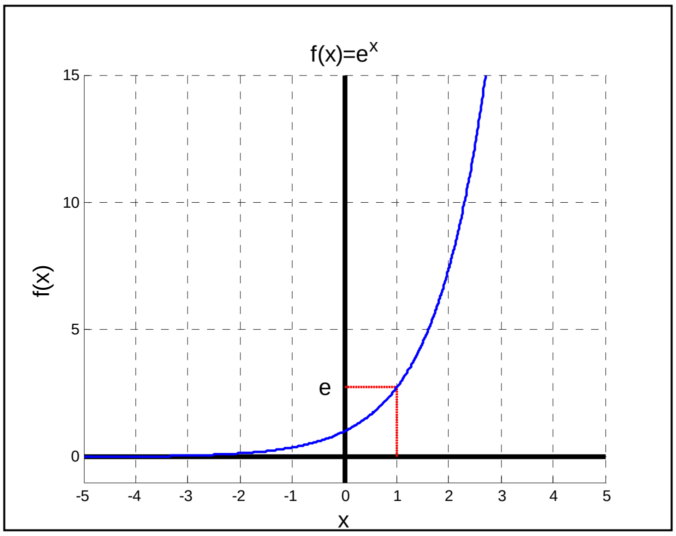

Figure 1.1. Exponential function.

– Figure 1.1. was created using following MATLAB code:

```
clear;

%DEFINE f(x).
e = 2.718281828;
x =[-5:0.01:5];
fx=e.^x;

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-5 5 -1 15]);
title('f(x)=e^x','FontSize',14); xlabel('x','FontSize',14);
ylabel('f(x)','FontSize',14);
line([-5 5],[0 0],'Color','k','LineWidth',3);
line([0 0],[-1 15],'Color','k','LineWidth',3);

%DRAW e.

text(-0.5,e,'e','FontSize',14)
line([0 1],[e e],'Color','r','LineWidth',2,'LineStyle','--');
line([1 1],[0 e],'Color','r','LineWidth',2,'LineStyle','--');

%DRAW f(x).
plot(x,fx,'LineWidth',2);
```

– The most important property of exponential function is that at any point it's derivative is the same as it's value:

$$(e^x)' = e^x$$

which will be proven in one of the next chapters.

2. Expressing exponential function as the sum of infinite series

– In this chapter we will show that exponential function can be expressed as the sum of infinite series like this [1]:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.1}$$

– We start with definition of Euler's number [3]:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{2.2}$$

and inserting it into (1.1) we get:

$$e^x = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^x$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} \tag{2.3}$$

– Introducing substitution:

$$m = nx$$

which gives:

$$n = \frac{m}{x}$$

$$\frac{1}{n} = \frac{x}{m}$$

into (2.3) we get:

$$e^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m \tag{2.4}$$

– Using binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \tag{2.5}$$

(2.4) can be written like:

$$e^x = \lim_{m\to\infty}\left[1 + m\frac{x}{m} + \frac{m(m-1)}{2!}\left(\frac{x}{m}\right)^2 + \frac{m(m-1)(m-2)}{3!}\left(\frac{x}{m}\right)^3 + \cdots + \left(\frac{x}{m}\right)\right]$$

$$= \lim_{m\to\infty}\left[1 + x + \frac{x^2}{2!}\frac{m(m-1)}{m^2} + \frac{x^3}{3!}\frac{m(m-1)(m-2)}{m^3} + \cdots + \left(\frac{x}{m}\right)\right] \tag{2.6}$$

– When m goes toward infinity (2.6) becomes:

$$e^x = 1 + x + \frac{x^2}{2!}\frac{m^2}{m^2} + \frac{x^3}{3!}\frac{m^3}{m^3} + \cdots + 0$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

3. Derivative of exponential function

– The most important attribute of exponential function is that it's derivative is also exponential function:

$$\frac{d}{dx}e^{x} = e^{x} \tag{3.1}$$

– Following chapters give two simiar proofs of this.

3.1. Proof of derivative of exponential function using rules for derivation

– Inserting (2.1) into (3.1) we get [1]:

$$\begin{aligned}\frac{d}{dx}e^{x} &= \frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\\ &= 0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\cdots\\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\\ &= e^{x}\end{aligned}$$

3.2. Proof of derivative of exponential function using definition of first derivative

– We will prove how first derivative looks like by starting with definition of first derivative:

$$\frac{d}{dx}f(x) = \lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx} \tag{3.2}$$

– Inserting (1.1) into (3.2) we get:

$$\begin{aligned}\frac{d}{dx}e^{x} &= \lim_{dx\to 0}\frac{e^{x+dx}-e^{x}}{dx}\\ &= \lim_{dx\to 0}\frac{e^{x}\left(e^{dx}-1\right)}{dx}\\ &= e^{x}\lim_{dx\to 0}\frac{e^{dx}-1}{dx} \qquad (3.3)\end{aligned}$$

– Inserting (2.1) into (3.3) we get:

$$\begin{aligned}\frac{d}{dx}e^{x} &= e^{x}\lim_{dx\to 0}\frac{1+dx+\frac{dx^2}{2!}+\frac{dx^3}{3!}+\cdots-1}{dx}\\ &= e^{x}\lim_{dx\to 0}\left(\frac{1}{dx}+\frac{dx}{dx}+\frac{1}{dx}\frac{dx^2}{2!}+\frac{1}{dx}\frac{dx^3}{3!}+\cdots-\frac{1}{dx}\right)\\ &= e^{x}\lim_{dx\to 0}\left(\frac{1}{dx}+1+\frac{dx}{2!}+\frac{dx^1}{3!}+\cdots-\frac{1}{dx}\right)\\ &= e^{x}\end{aligned}$$

4. References

[1] http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/Exponential_Function.htm

[2] http://planetmath.org/encyclopedia/DerivativeOfExponentialFunction.html

[3] e - Euler's number.doc