1. Logarithms

- Idea behind logarithms is to express each number 'x' as exponent 'u' to some base number 'b':

$$\log_b x = u \quad \Leftrightarrow \quad x = b^u \tag{1.1}$$

- This way each number 'x' is mapped to some other number 'u'.
- This makes multiplication of two numbers 'x' and 'y' extremly easy using tables which contain those mappings:

$$xy = b^u b^v = b^{u+v}$$

$$\log_b(xy) = u + v$$

- As we can see, instead of multplying 'x' and 'y', which is extremly time consuming, we only need to add 'u' and 'v' to which 'x' and 'y' are mapped.
- To get 'xy' we simply need to use logarithm tables to see which number is mapped to 'u+v'.

- Invention of Logarithms in 17th century boosted science as much as computers did in 20th century [1].
- Example of how they were used can be found at the end of this tutorial.

2. Properties of logarithms

- In following chapters we shall prove properties of logarithms which made them so popular:

$$\log_b(xy) = \log_b(x) + \log_b(y) \tag{2.1}$$

$$\log_b(x^y) = y\log_b(x) \tag{2.2}$$

$$\log_b(\sqrt[y]{x}) = \frac{1}{y}\log_b(x) \tag{2.3}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \tag{2.4}$$

- The above properties of logarithms will be proven in following chapters using logarithm definition (1.1).

- Another way to prove those properties would be to use logarithm definition (1.1) to prove (2.1).

- Then we can use (2.1) to prove (2.2):

$$\log_b(x^y) = \log_b(\underbrace{x \cdot x \cdots x}_{y}) = \underbrace{\log_b(x) + \log_b(x) + \cdots + \log_b(x)}_{y} = y\log_b(x) \tag{2.5}$$

- We can use (2.2) to prove (2.3):

$$\log_b(\sqrt[y]{x}) = \log_b\left(x^{1/y}\right) = \frac{1}{y}\log_b(x) \tag{2.6}$$

- We can use (2.1) and then (2.2) to prove (2.4):

$$\log_b\left(\frac{x}{y}\right) = \log_b\left(xy^{-1}\right) = \log_b(x) + \log_b\left(y^{-1}\right) = \log_b(x) - \log_b(y) \tag{2.7}$$

2.1. Proof that logarithms transform product to sum

– In this chapter we shall prove relation (2.1) [3] which is:

$$\log_b(xy) = \log_b(x) + \log_b(y) \tag{2.8}$$

– To do so we shall introduce following variables:

$$u = \log_b x \tag{2.9}$$

$$v = \log_b y \tag{2.10}$$

– Using logarithm definition (1.1), equations (2.9) and (2.10) become:

$$x = b^u \tag{2.11}$$

$$y = b^v \tag{2.12}$$

– Multiplying (2.11) with (2.12) we get:

$$\begin{aligned} xy &= b^u b^v \\ &= b^{u+v} \end{aligned} \tag{2.13}$$

– Once again using logarithm definition (1.1) on (2.13) we get:

$$\log_b(xy) = u + v \tag{2.14}$$

– Inserting (2.9) and (2.10) into (2.14) we get (2.1) which is exactly what we wanted to prove.

2.2. Proof that logarithms transform qotients to differences

– In this chapter we shall prove relation (2.4) [3] which is:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \tag{2.15}$$

– Dividing (2.11) by (2.12) we get:

$$\begin{aligned} \frac{x}{y} &= \frac{b^u}{b^v} \\ &= b^{u-v} \end{aligned} \tag{2.16}$$

– Once again using logarithm definition (1.1) on (2.16) we get:

$$\log_b(xy) = u + v \tag{2.17}$$

– Inserting (2.9) and (2.10) into (2.17) we get (2.4) which is exactly what we wanted to prove.

2.3. Proof that logarithms transform qotients to differences

– In this chapter we shall prove relation (2.2) [3] which is:

$$\log_b(x^y) = y\log_b(x) \tag{2.18}$$

– Rising (2.11) to the potention of y we get:

$$x = b^u \quad \Big/ \uparrow^y$$

$$x^y = \left(b^u\right)^y$$

$$x^y = b^{yu} \quad \Big/ \log_b$$

$$\log_b x^y = \log_b b^{yu} \tag{2.19}$$

– Using logarithm definition (1.1), (2.19) becomes:

$$\log_b x^y = yu \tag{2.20}$$

– Inserting (2.9) into (2.20) we get (2.18) which we wanted to prove.

3. Example of using logarithms

– We will now explain how logarithms were used [2] by trying to find solution to:

$$x = 630 \cdot 4.1$$

$$x = 63 \cdot 10 \cdot 41 \cdot 10^{-1} \Big/ \log_{10}$$

– What we do first is to take logarithm of the both sides of the equation:

$$\begin{aligned} \log_{10} x &= \log_{10}\left(63 \cdot 10 \cdot 41 \cdot 10^{-1}\right) \\ &= \log_{10} 63 + \log_{10} 10 + \log_{10} 41 + \log_{10} 10^{-1} \end{aligned} \quad (3.1)$$

– Then we have to look up values for the two logarithms on te right using following table:

N	$\log_{10}N$
25	1.3979
26	1.4150
41	1.6128
63	1.7993

Table 3.1. Logarithms table.

– Using logarithms table 3.1., equation (3.1) can be written like:

$$\begin{aligned} \log_{10} x &= 1.7993 + 1 + +1.6128 + -1 \\ &= 3.4121 \end{aligned}$$

– And now to find x we simply have to go into oposite direction:

$$\begin{aligned} \log_{10} x &= 2 + 1.4121 \\ &= \log_{10} 100 + \log_{10} ? \end{aligned}$$

– Using table 3.1. we can see that number whose $\log_{10}$ is 1.4121 lies between 25 and 26 so we choose 25.5:

$$\log_{10} x \approx \log_{10} 100 + \log_{10} 25.5$$

$$x \approx 100 \cdot 25.5 = 2550$$

4. References

[1] http://en.wikipedia.org/wiki/Logarithm#History

[2] http://www.krysstal.com/logarithms.html

[3] http://campus.northpark.edu/math/precalculus/Transcendental/Logarithmic/Properties/index.html