Series

1. Introduction

– Series is sequence of terms with addition operations between them, for instance [1]:

$$1 + 2 + 3 + 4 + 5 + ... + 99 + 100 \tag{1.1}$$

– Terms of the sequence are produced according to a certain rule allowing us to represent the whole series in a compact form using sumation sign $\sum$ like this:

$$\sum_{k=1}^{100} k \tag{1.2}$$

– Expressions (1.1) and (1.2) are equivalent, they are just two different ways of expressing the same thing, so we can write:

$$\sum_{k=1}^{100} k = 1 + 2 + 3 + 4 + 5 + ... + 99 + 100 \tag{1.3}$$

2. Convergence

– If sum of series converges to some number, as number of term goes into infinity, then it is said that series converges to that value.

2.1. Convergence of geometric series [4]

– Geometric series is defined like this:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \tag{2.1}$$

– Convergence of geometric series can be found like this:

$$\begin{aligned} S_n &= 1 + x + x^2 + x^3 + \cdots + x^n \quad \Big/ \; x \\ - \quad xS_n &= \quad x + x^2 + x^3 + \cdots + x^n + x^{n+1} \\ \hline (1-x)S_n &= 1 - x^{n+1} \end{aligned}$$

$$S_n = \frac{1 - x^{n+1}}{1 - x} \tag{2.2}$$

– For |x|<1 as n goes toward infinity second term in numerator converges to zero so sum converges to following value:

$$S_n = \frac{1}{1-x} \quad , \quad |x|<1 \tag{2.3}$$

– Combining (2.1) with (2.3) we can write:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n \quad , \quad |x|<1 \tag{2.4}$$

– Equation (2.4) shown that we were able to approximate expresion on the left side around |x|>1 with the sum on the right:

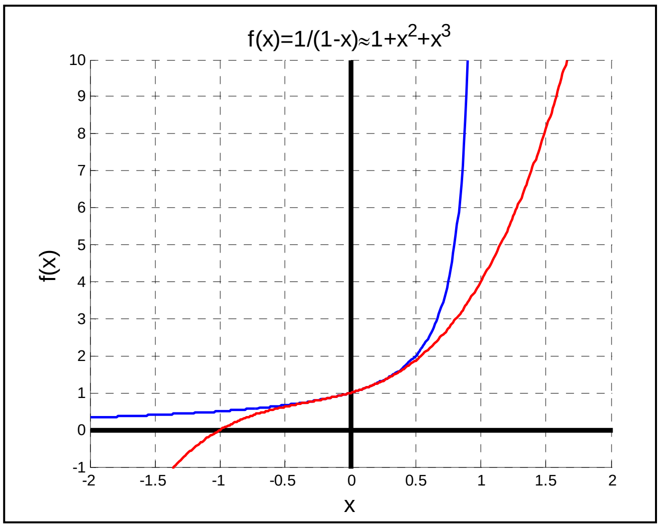


Figure 2.1. Approximation using polynomials.

2.2. Convergence of harmonic series [4]

– Harmonic series is defined like this:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{2.5}$$

– We will find convergence by comparing series with area under function 1/x which is defined using integral like this:

$$\int_1^{\infty} \frac{1}{x} dx = \ln x \Big/_1^{\infty} = \ln \infty - \ln 1 = \infty - 0 = \infty \tag{2.6}$$

– Such comparison is ilustrated on following figure:

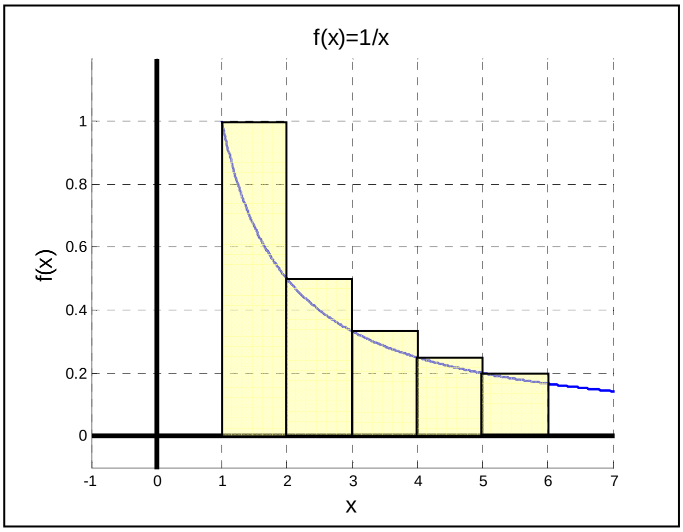


Figure 2.2. Comparing harmonic series with area under function 1/x.

– Each yellow square on figure Figure 2.2. represents one term of harmonic series.
– First square has surface 1·1=1 which is value of the first term of harmonic series.
– Second square has surface 1·1/2=1/2 which is value of the second term of harmonic series.
– Third square has surface 1·1/3=1/3 which is value of the third term of harmonic series and so on.

– Figure Figure 2.2. shown that sum of all squares, which represent terms of harmonic series, is greater then area under function 1/x.
– And since we have calculated with (2.6) that area under 1/x is infinite that means that harmonic series also goes into infinity and does not converge.

2.3. Convergence of series with factorials [4]

– We will now try to determine convergence of following series:

$$\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots \tag{2.7}$$

$$= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots \tag{2.8}$$

– Convergence will be proven using following observation:

$$\frac{1}{k!}=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdots k}=\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdots\frac{1}{k}\le 1\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}=\frac{1}{2^{k-1}} \tag{2.9}$$

– Using (2.9), (2.7) can be written like this:

$$\sum_{k=0}^{\infty}\frac{1}{k!} \le \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2^{2-1}}+\frac{1}{2^{3-1}}+\frac{1}{2^{4-1}}+\cdots$$

$$\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

$$\le 1+\left[\left(\frac{1}{2}\right)^0+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\right] \tag{2.10}$$

– Term inside square brackets is geometric series as defined with (2.1).
– Since $|x|=|1/2|<1$ this term converges as defined by (2.3) so we can write:

$$\sum_{k=0}^{\infty}\frac{1}{k!} \le 1+\frac{1}{1-\frac{1}{2}}=1+\frac{1}{\frac{2}{2}-\frac{1}{2}}=1+\frac{1}{\frac{1}{2}}=1+2=3 \tag{2.11}$$

– This shows that series converges to a number smaller then 3.
– It will be hown later that this number is acctually e.

3. Power series

– Power series is each infinite series of the following form [2]:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \qquad (3.1)$$

where: a_n – coefficient of the n-th term,
c – constant around which series is centered,
x – value close to c.

3.1. Usage

– Power series is one of the greatest weapons available to mathematicians.
– Power series allow us to compare functions by transforming each of them into power series.
– One of the greatest results of such aproach is well known Euler's functions:

$$e^{ix} = \cos(x) + i\sin(x)$$

which was obtained by trasforming each of the functions into their power series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

– Alos by transforming function around value of interest they simplify our calculations.

3.2. Idea

– Power function is defined like this:

$$f(x) = x^n \tag{3.2}$$

where: f(x) – power function,
n – non negative integer 0,1,2,3,...

– Idea to use power series to aproximate functions comes from properties of power functions [3]:
– Each power function has dead-zone, symetric interval around x=0 to which it has negligilbe influence.
– While approximating you are adding more power functions and each should have dead-zone around interval where approximation is already good, and it should improve approximation only outside that interval.
– Power functions are also very easy to differentiate and integrate.

– Other properties of power function include:
  – for -1<x<1 the power functions with smaller n have bigger values, they are more dominant in this reagon
  – for x<-1 and 1<x the power functions with larger n have bigger values, they are more dominant in this reagon
  – each power function goes through point (1,1)
  – for odd n power function is also odd and goes through point (-1,-1)
  – for even n power function is also even and goes through point (-1,1)

– These properties are ilustrated on following figure:

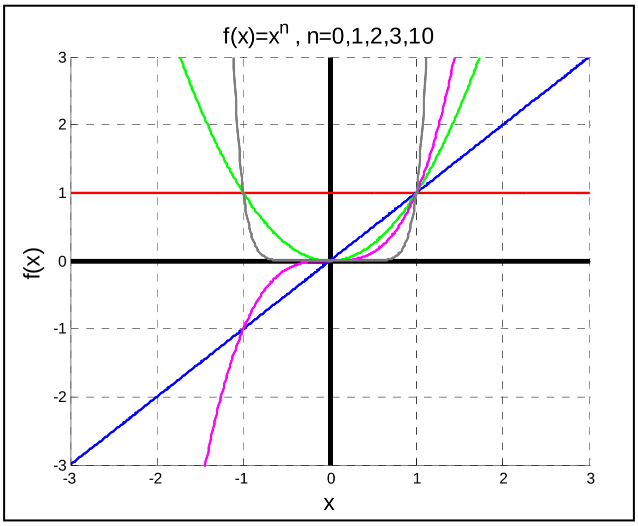


Figure 3.1. Power functions for n=0,1,2,3,10

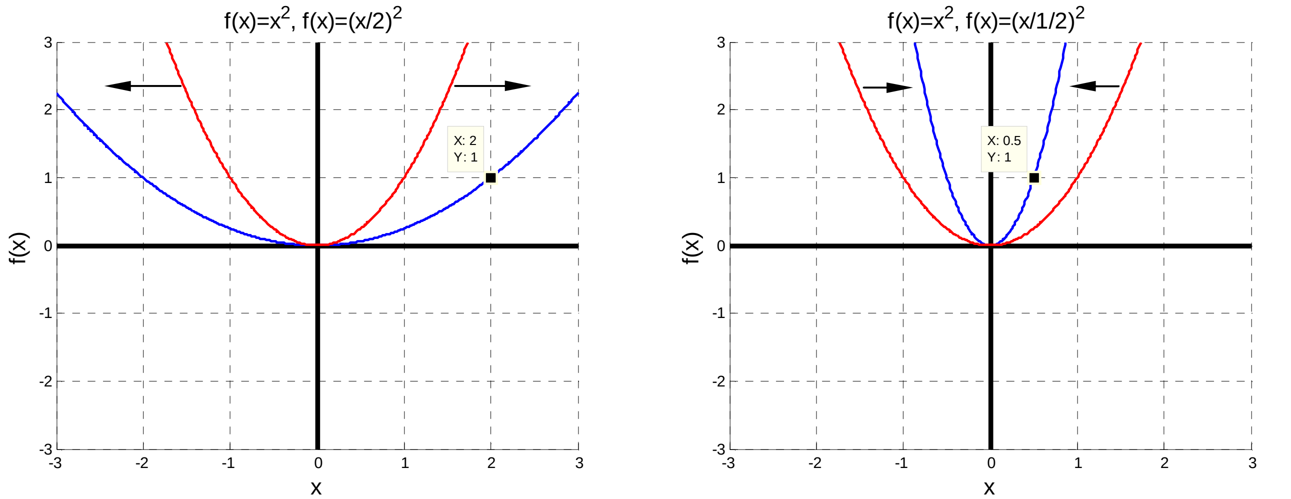
3.3. Scaling

– In order to acctually be able to aproximate anything with (3.2) we need to introduce scaling factor:

$$f(x) = \left(\frac{x}{b}\right)^n = ax^n \tag{3.3}$$

where: a,b – scaling factors.

– If you want your function to be twice as wide then simple use b=2 as shown on figure Figure 3.2.a).
– If you want your function to be two times narrower then simple use b=1/2 as shown on figure Figure 3.2.b).



a) Stretching function. b) Schrinking function.

Figure 3.2. Stretching and shrinking function using scaling factor.

– When |a|<1 power function stretches decreasing it's dominance around x=0.
– When 1<|a| power function shrinks increasing it's dominance around x=0.

– Following figure ilustrates how 'a' scaling factor changes power function:

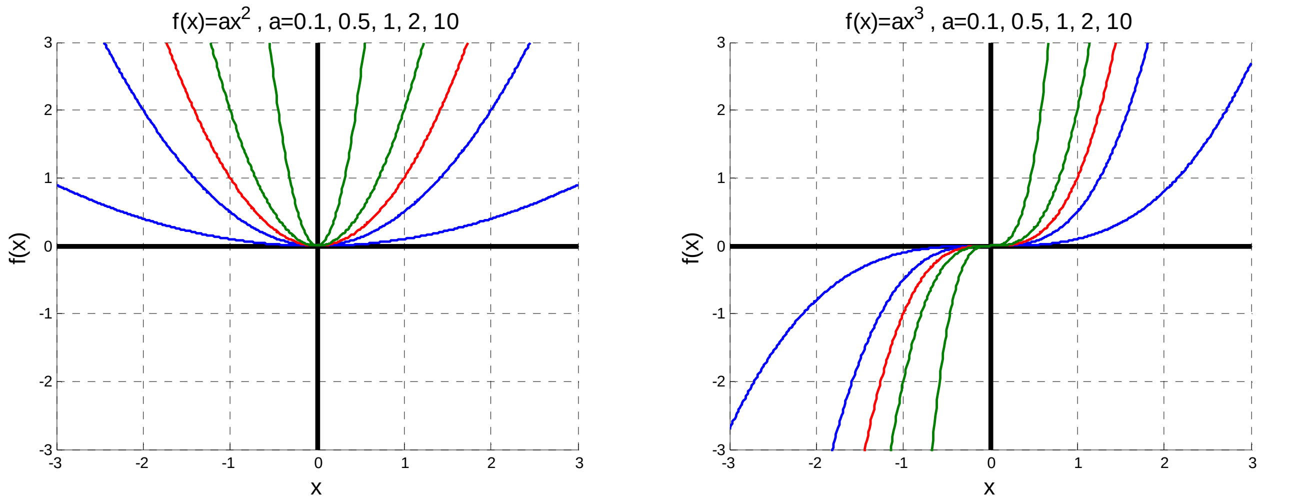


Figure 3.3. Influence on scaling factor.

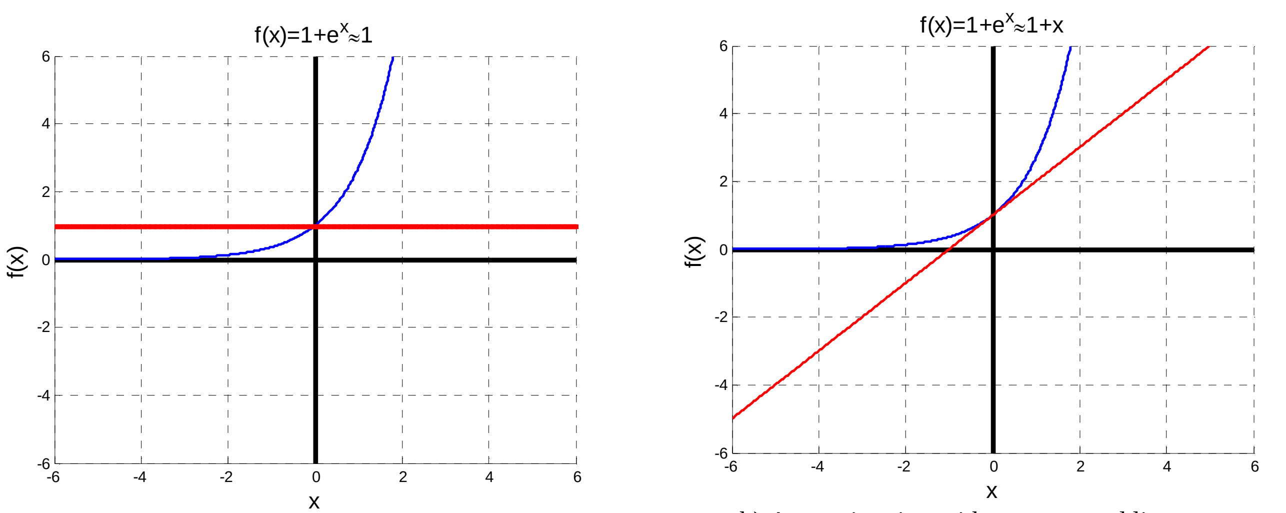
3.4. Approximating

– As said before our goal is to approximate function using series of power functions like this:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

– We will now try to build our series term by term by choosing adequate scaling factors in order to approximate $f(x)=e^x$.

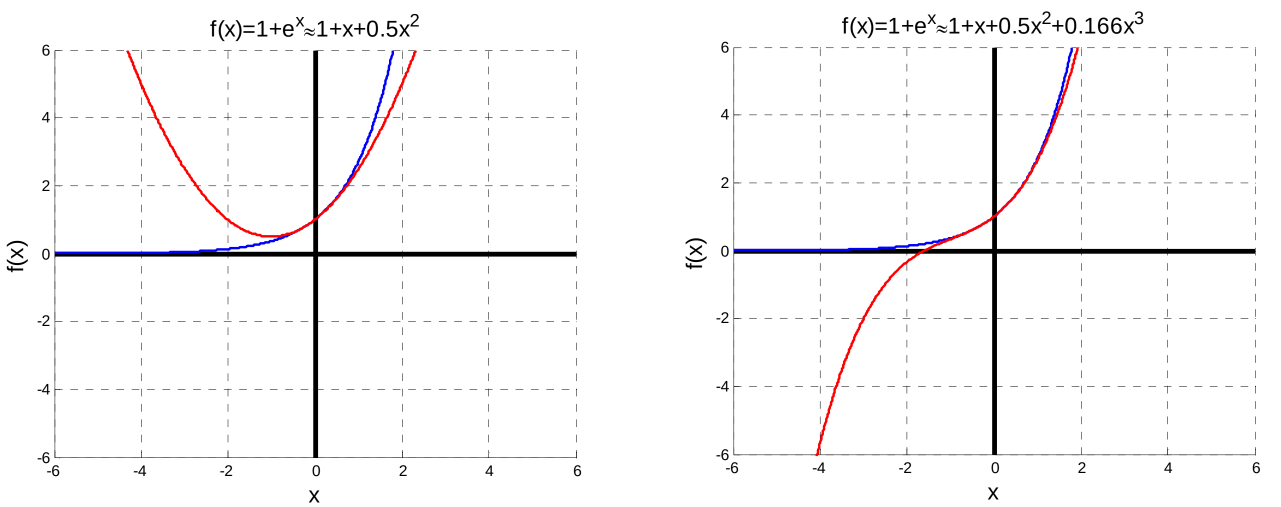
– First is the constant term which ensures that the function and the series agree at the point x=0 as shown on Figure 3.4.a).
– After that we add in the linear term figure Figure 3.4.b).
– These first two terms together give so called linear approximation of function f(x) which is better as we are closer to x=0.



a) Approximation with constant term.
b) Approximation with constant and linear term.

Figure 3.4. Approximation with constant and linear terms.

– Now we would like to add in another term which will increase the agreement between the series and the function. However, we don't want to mess up what we've already accomplished with the constant and linear term.
– We can do this with a quadratic term.
Near the origin, quadratic term is less dominant than constant and linear so we won't destroy the agreement we have so far.
Away from the origin, quadratic term will be dominant allowing us to fine-tune the approximation away from x=0.
– With the appropriate choice of scaling coefficient, the quadratic term will become dominant just at the point where we need to increase the extent to which the two functions agree.
– Using this kind of logic introduction of quadratic and cubic terms gives results as shown on following figure:



a) Approximation with 3 terms.
b) Approximation with 4 terms.

Figure 3.5. Approximation with 3 and 4 terms.

4. Taylor's series

- Taylor's series is approximation of f(x) around x=c with power series using functions's derivates at x=c to calculate coefficients:

$$\begin{aligned} f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \Big/ \frac{d}{dx} \\ f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \Big/ \frac{d}{dx} \\ f''(x) &= 2a_2 + 3\cdot 2a_3(x-c) + \cdots \Big/ \frac{d}{dx} \\ f'''(x) &= 3\cdot 2a_3 + \cdots \Big/ \frac{d}{dx} \end{aligned} \tag{4.1}$$

- Inserting x=c into each of those derivation we get values for coefficients a_n:

$$\begin{aligned} f(c) &= a_0 \\ f'(c) &= a_1 \\ f''(c) &= 2a_2 \\ f'''(c) &= 3\cdot 2a_3 \\ f^{(4)}(c) &= 4\cdot 3\cdot 2a_4 \end{aligned}$$

- Inserting calculated coefficients into definition for power series (4.1) we get:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \frac{f^{(4)}(c)}{4\cdot 3\cdot 2}(x-c)^4 + \cdots$$

$$f(x) = f(c) + f'(c)\frac{(x-c)}{1!} + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + f^{(4)}(c)\frac{(x-c)^4}{4!} + \cdots$$

5. MacLaurin's series

– MacLaurin's series is Taylor's series when c=0:

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \cdots \quad (5.1)$$

5.1. MacLaurin's series of function sin(x)

– Having in mind that:

$$\sin' x = \cos x$$

$$\cos' x = -\sin x$$

MacLaurin's series of function sin(x) can be calculated using (5.1) like this:

$$\begin{aligned}\sin(x) &= \sin 0 + \cos 0\frac{x}{1!} - \sin 0\frac{x^2}{2!} - \cos 0\frac{x^3}{3!} + \sin 0\frac{x^4}{4!} + \cos 0\frac{x^5}{5!} - \sin 0\frac{x^6}{6!} - \cos 0\frac{x^7}{7!} + \cdots \\ &= 0 + 1\frac{x}{1!} - 0\frac{x^2}{2!} - 1\frac{x^3}{3!} + 0\frac{x^4}{4!} + 1\frac{x^5}{5!} - 0\frac{x^6}{6!} - 1\frac{x^7}{7!} + \cdots \\ &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \quad (5.2)\end{aligned}$$

– Following figure shows successive approximation of sin(x) with macLaurin's series as k=0,1,2,...:

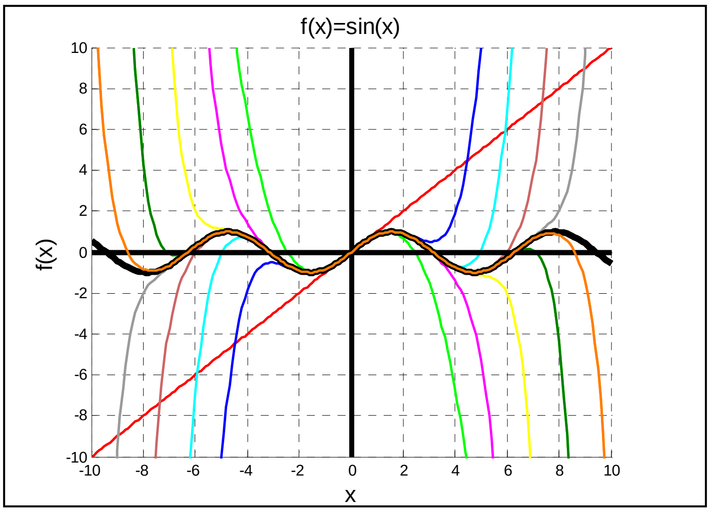


Figure 5.1. Succesive aproximation of sin(x) with MacLaurin's series.

5.2. MacLaurin's series of function cox(x)

– MacLaurin's series of function cos(x) can be calculated using (5.1) like:

$$\cos(x) = \cos 0 - \sin 0\frac{x}{1!} - \cos 0\frac{x^2}{2!} + \sin\frac{x^3}{3!} + \cos 0\frac{x^4}{4!} - \sin 0\frac{x^5}{5!} - \cos 0\frac{x^6}{6!} + \sin 0\frac{x^7}{7!} + \cdots$$

$$= 1 - 0\frac{x}{1!} - 1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} - 0\frac{x^5}{5!} - 1\frac{x^6}{6!} + 0\frac{x^7}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \qquad (5.3)$$

– Following figure shows successive approximation of cos(x) with macLaurin's series as k=0,1,2,...:

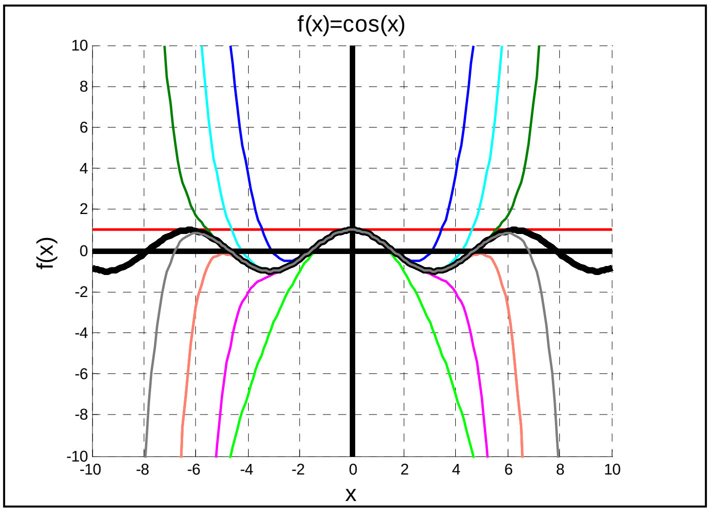


Figure 5.2. Succesive aproximation of cos(x) with MacLaurin's series.

5.3. MacLaurin's series of function e^x

– Having in mind that:

$$\left(e^x\right)' = \left(e^x\right)'' = \cdots = e^x$$

MacLaurin's series of function e^x can be calculated using (5.1) like:

$$\begin{aligned} e^x &= e^0 + e^0\frac{x}{1!} + e^0\frac{x^2}{2!} + e^0\frac{x^3}{3!} + e^0\frac{x^4}{4!} + e^0\frac{x^5}{5!} + e^0\frac{x^6}{6!} + e^0\frac{x^7}{7!} + \cdots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\ &= \sum_{k=0}^{\infty}\frac{x^k}{k!} \end{aligned} \quad (5.4)$$

– Following figure shows successive approximation of e^x with macLaurin's series as k=0,1,2,...:

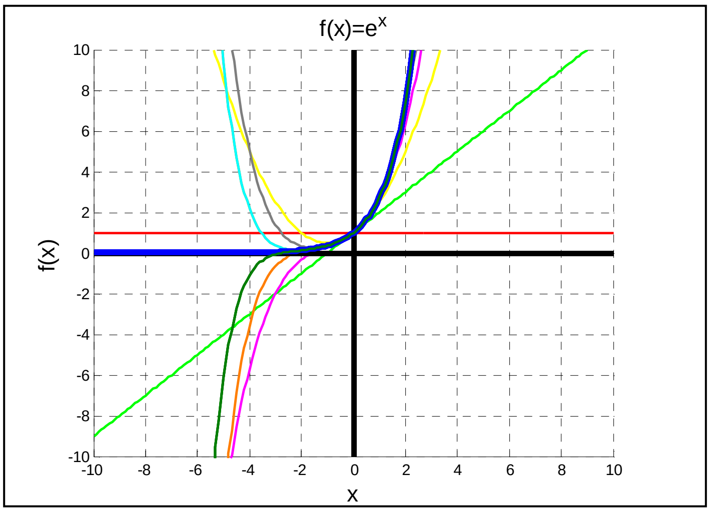


Figure 5.3. Succesive aproximation of e^x with MacLaurin's series.

5.4. MacLaurin's series of function e^{ix}

– MacLaurin's series of function e^{ix} can be calculated using (5.4) like:

$$\begin{aligned} e^{ix} &= 1+\frac{ix}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+\cdots \\ &= 1+i\frac{x}{1!}-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\frac{x^6}{6!}-i\frac{x^7}{7!}+\cdots \\ &= \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+\left(i\frac{x}{1!}-i\frac{x^3}{3!}+i\frac{x^5}{5!}-i\frac{x^7}{7!}+\cdots\right) \end{aligned} \tag{5.5}$$

– Inserting (5.3) and (5.2) we get:

$$e^{ix} = \cos x + i\sin x$$

6. MATLAB code

6.1. Figure 3.1.

```
clear;
w=3;
x =[-w:0.01:w];
n=4;
f=x.^n;

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-w w -w w]);
title('f(x)=x^n , n=0,1,2,3,10','FontSize',14);
xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW f(x).
n=0;  f=x.^n; plot(x,f,'Color','r','LineWidth',2);
n=1;  f=x.^n; plot(x,f,'Color','b','LineWidth',2);
n=2;  f=x.^n; plot(x,f,'Color','g','LineWidth',2);
n=3;  f=x.^n; plot(x,f,'Color','m','LineWidth',2);
n=10; f=x.^n; plot(x,f,'Color',[0.5 0.5 0.5],'LineWidth',2);
```

6.2. Figure 3.3.

```
clear;
w=3;
x =[-w:0.01:w];
a=1;
n=3;
f=a*x.^n;

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-w w -w w]);
title('f(x)=ax^3 , a=0.1, 0.5, 1, 2, 10','FontSize',14);
xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW f(x).
a=0.1; f=a*x.^n; plot(x,f,'Color','b','LineWidth',2);
a=0.5; f=a*x.^n; plot(x,f,'Color','b','LineWidth',2);
a=1;   f=a*x.^n; plot(x,f,'Color','r','LineWidth',2);
a=2;   f=a*x.^n; plot(x,f,'Color',[0 0.5 0],'LineWidth',2);
a=10;  f=a*x.^n; plot(x,f,'Color',[0 0.5 0],'LineWidth',2);
```

6.3. Figure 3.4.

```
clear;
w=6;
x =[-w:0.01:w];
e = 2.718281828;
sine=e.^x;
a=1;
n=3;
f=a*x.^n;

kmax = 4;
k    = [0:kmax];
ak   = 1./factorial(k)

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-w w -w w]);
title('f(x)=1+e^x\approx1+x+0.5x^2+0.166x^3','FontSize',14); xlabel('x','FontSize',14);
ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW f(x).
f=ak(1);
f=f+ak(2)*x;
f=f+ak(3)*x.^2;
f=f+ak(4)*x.^3;
plot(x,f,'Color','r','LineWidth',2);
plot(x,sine,'Color','b','LineWidth',2);
```

6.4. Figure 5.1.

```
clear;
kmax = 9;
w    = 10;
x    = [-w:0.1:w];
fx   = sin(x);
k    = [0:kmax];
ak   = (-1).^k;
ak   = ak./factorial(2*k+1)
fk   = zeros(length(k),length(x));
for i=1:length(ak)
  fk(i,:)=ak(i).*x.^(2*k(i)+1);
end

fk_size=size(fk);
if fk_size(1,1)==1
  f=fk;
else
  f=sum(fk);
end

%PREPARE FIGURE.
figure(1); grid on; hold on; axis([-w w -w w]);
title('f(x)=sin(x)','FontSize',14);
xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW FUNCTIONS.
plot(x,fx,'Color','k','LineWidth',4);
plot(x,f,'Color',[1 0.5 0],'LineWidth',2);
```

6.5. Figure 5.2.

```
clear;
kmax = 7;
w    = 10;
x    = [-w:0.1:w];
fx   = cos(x);
k    = [0:kmax];
ak   = (-1).^k;
ak   = ak./factorial(2*k)
fk   = zeros(length(k),length(x));
for i=1:length(ak)
  fk(i,:)=ak(i).*x.^(2*k(i));
end

fk_size=size(fk);
if fk_size(1,1)==1
  f=fk;
else
  f=sum(fk);
end

%PREPARE FIGURE.
figure(1); grid on; hold on; axis([-w w -w w]);
title('f(x)=cos(x)','FontSize',14); xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW FUNCTIONS.
plot(x,fx,'Color','k','LineWidth',4);
plot(x,f,'Color',[0.5 0.5 0.5],'LineWidth',2);
```

6.6. Figure 5.3.

```
clear;
kmax = 7;
w    = 10;
x    = [-w:0.1:w];
e    = 2.718281828;
fx   = e.^x;
k    = [0:kmax];
ak   = 1./factorial(k)
fk   = zeros(length(k),length(x));
for i=1:length(ak)
  fk(i,:)=ak(i).*x.^k(i);
end

fk_size=size(fk);
if fk_size(1,1)==1
  f=fk;
else
  f=sum(fk);
end

%PREPARE FIGURE.
figure(1); grid on; hold on; axis([-w w -w w]);
title('f(x)=e^x','FontSize',14); xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW FUNCTIONS.
plot(x,fx,'Color','b','LineWidth',4);
plot(x,f,'Color',[0 0.5 0],'LineWidth',2);
```

6.7. Figure 2.1.

```
clear;
w=10;
x =[-2:0.01:0.999];
r =[-2:0.01:2];
f=1./(1-x);

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-2 2 -1 10]);
title('f(x)=1/(1-x)\approx1+x^2+x^3','FontSize',14);
xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW f(x).
plot(x,f,'Color','b','LineWidth',2);
plot(r,1+r+r.^2+r.^3,'Color','r','LineWidth',2);
```

6.8. Figure 2.2.

```
clear;
w=10;
x =[1:0.01:8];
series =[1 1/2 1/3 1/4 1/5 1/6];
k=[1 2 3 4 5 6];
f=1./x;

%PREPARE FIGURE.
figure(1); clf; grid on; hold on; axis([-1 7 -0.1 1.2]);
title('f(x)=1/x','FontSize',14);
xlabel('x','FontSize',14); ylabel('f(x)','FontSize',14);
line([-w w],[0 0],'Color','k','LineWidth',3);
line([0 0],[-w w],'Color','k','LineWidth',3);

%DRAW f(x).
plot(x,f,'Color','b','LineWidth',2);
```

7. References

[1] http://en.wikipedia.org/wiki/Series_(mathematics)

[2] http://en.wikipedia.org/wiki/Power_series

[3] http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/series/powers.html

[4] http://www.ugrad.math.ubc.ca/coursedoc/math101/notes/series/convergence.html