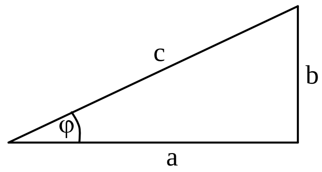
Trigonometric Functions

1. Introduction

– Basic trigonometric functions are called 'sine' and 'cosine' and they are defined using square triangel for |φ|<π/2:

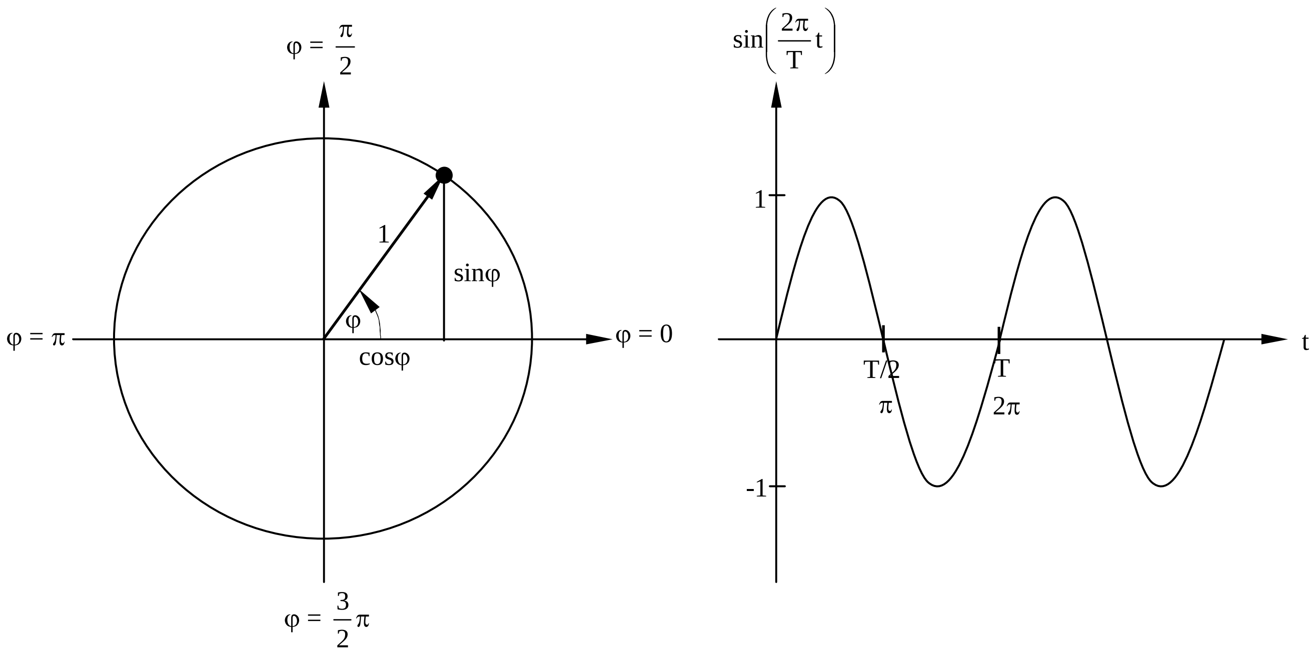


$$\sin\varphi = \frac{b}{c}$$

$$\cos\varphi = \frac{a}{c}$$

Figure 1.1. Definition of 'sine' and 'cosine' functions using square triangel.

– More general definition uses unit circle and expands definitions for all φ:



a) Definition of 'sine' function using unit circle. b) Projection of vector to horizontal axis.

Figure 1.2. Definition of 'sine' and 'cosine' functions using unit circle.

– Using unit circle, sine function is defined as projection of rotating vector of unit length on horizontal axis.
– Using unit circle, cosine function is defined as projection of rotating vector of unit length on vertical axis.
– As vector rotates in counter-clockwise direction it's projection on horizontal axis changes as shown on 1.2.b).

2. List of trigonometric functions

– Since sine and cosine functions are defined as projections on vertical and horizontal axis respectivly following rules apply:

$$\sin(-\varphi) = -\sin(\varphi) \tag{2.1}$$

$$\cos(-\varphi) = \cos(\varphi) \tag{2.2}$$

– Derivatives fo sine and cosine function have following properties:

$$\sin'(\varphi) = \cos(\varphi) \tag{2.3}$$

$$\cos'(\varphi) = -\sin(\varphi) \tag{2.4}$$

– Following trigonometric function is derived from Pythagorean theorem:

$$1 = \cos^2(\varphi)+\sin^2(\varphi) \tag{2.5}$$

– Euler's function, derived from Taylor's series, is the most important trigonometric function:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \tag{2.6}$$

– Euler's function can be used to prove following trigonometric functions:

$$e^{j\varphi} + e^{-j\varphi} = 2\cos(\varphi) \tag{2.7}$$

– Euler's function can also be used to prove following two trigonometric functions:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{2.8}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{2.9}$$

– Trigonometric functions (2.8) and (2.9) can be used to prove following three trigonometric functions:

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \tag{2.10}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \tag{2.11}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \tag{2.12}$$

– Trigonometric functions (2.8) and (2.9) can also be used to prove following two trigonometric functions:

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{2.13}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{2.14}$$

– Last two trigonometric functions can be used to prove following two trigonometric functions:

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{2.15}$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{2.16}$$

– Other trigonometric functions include:

$$\int_{-\pi}^{\pi} \cos(k\varphi) d(\varphi) = 0 \quad , \quad k \neq 0 \tag{2.17}$$

$$\int_{-\pi}^{\pi} \sin(k\varphi) d(\varphi) = 0 \quad , \quad k \neq 0 \tag{2.18}$$

$$\int_{0}^{2\pi} \cos^2(\varphi) d\varphi = \begin{cases} 2\pi & \text{for } \varphi = 0 \\ \pi & \text{for } \varphi \neq 0 \end{cases} \tag{2.19}$$

$$\int_{0}^{2\pi} \sin^2(\varphi) d\varphi = \begin{cases} 0 & \text{for } \varphi = 0 \\ \pi & \text{for } \varphi \neq 0 \end{cases} \tag{2.20}$$

– Trigonometric functions for descrete signals:

$$\sum_{t=0}^{N-1} \cos[(k\omega t] = 0 \quad , \quad k \neq 0 \tag{2.21}$$

$$\sum_{t=0}^{N-1} \sin[(k\omega t] = 0 \quad , \quad k \neq 0 \tag{2.22}$$

$$\sum_{t=0}^{N-1} \cos^2(k\omega t) = \begin{cases} N & \text{for } k = 0 \\ \frac{N}{2} & \text{for } k \neq 0 \end{cases} \tag{2.23}$$

$$\sum_{t=0}^{N-1} \sin^2(k\omega t) = \begin{cases} 0 & \text{for } k = 0 \\ \frac{N}{2} & \text{for } k \neq 0 \end{cases} \tag{2.24}$$

3. Proof

– We shall now prove following two relations:

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \tag{2.25}$$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{2.26}$$

– We start by using Euler's function in following form:

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta)+i\sin(\alpha+\beta) \tag{2.27}$$

– Left side of (2.25) can also be witten like this:

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha+i\beta} \\ &= e^{i\alpha}e^{i\beta} \\ &= (\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta) \\ &= \cos\alpha\cos\beta+i\sin\alpha\cos\beta+i\cos\alpha\sin\beta-\sin\alpha\sin\beta \\ &= (\cos\alpha\cos\beta-\sin\alpha\sin\beta)+i(\sin\alpha\cos\beta+i\cos\alpha\sin\beta) \end{aligned}$$

– Since left sides of (2.25) and (1.10) are equal this means that right sides must also be equal and this is true only if equations (2.23) and (2.24) true.

4. Proof

– Once again we shall prove following relation:

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{2.28}$$

– Relation will be proven using following figure:

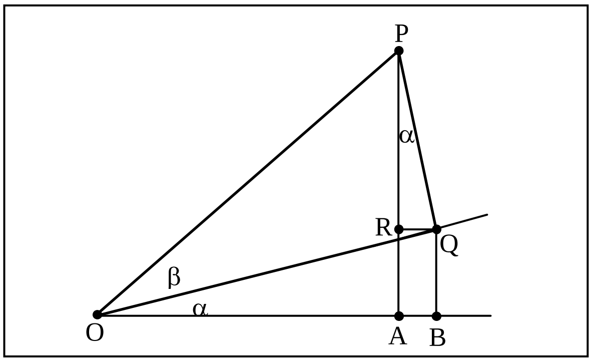


Figure 2.1. Proof.

– Angle ∠RPQ=α because:

$$\angle OPQ = 90^o - \beta \tag{2.29}$$

$$\angle OPA = 90^o - (\alpha+\beta) \tag{2.30}$$

$$\angle RPQ = \angle OPQ - \angle OPA \tag{2.31}$$

– Inserting (2.27) and (2.28) into (2.29) we get:

$$\begin{aligned}\angle RPQ &= 90^0 - \beta - [90^0 - (\alpha+\beta)]\\ &= 90^0 - \beta - 90^0 + (\alpha+\beta)\\ &= \alpha\end{aligned}$$

– We continue by expressing left side of expression (2.26) like this:

$$\begin{aligned}\sin(\alpha+\beta) &= PR + RA\\ &= PR + QB\end{aligned} \tag{2.32}$$

– Then we use following relations to express BQ and RP:

$$\cos\alpha = \frac{QB}{QO} \Rightarrow QB = \cos\alpha \cdot QO \tag{2.33}$$

$$\sin\alpha = \frac{PR}{PQ} \Rightarrow PR = \sin\alpha \cdot PQ \tag{2.34}$$

– Inserting (2.31) and (2.32) into (2.30) we get:

$$\sin(\alpha+\beta) = \sin\alpha \cdot PQ + \cos\alpha \cdot QO \tag{2.35}$$

– Now we can use following relations to express OQ and PQ:

$$\cos\beta = PQ \tag{2.36}$$

$$\sin\beta = QO \tag{2.37}$$

– Inserting (2.34) and (2.35) into (2.33) we finally get (2.26) which we wanted to prove.

4.1. Proof

– We shall now prove following relation:

$$\sin'(x) = \cos(x) \tag{2.38}$$

– Using definition for first derivative [1]:

$$\frac{d}{dx}f(x) = \lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx}$$

we can write:

$$\frac{d}{dx}\sin(x) = \lim_{dx\to 0}\frac{\sin(x+dx)-\sin(x)}{dx}$$

– Using (2.8) we can write:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{dx\to 0}\frac{\sin(x)\cos(dx)+\cos(x)\sin(dx)-\sin(x)}{dx} \\
&= \lim_{dx\to 0}\frac{\sin(x)\cos(dx)-\sin(x)}{dx} + \lim_{dx\to 0}\frac{\cos(x)\sin(dx)}{dx} \\
&= \sin(x)\lim_{dx\to 0}\frac{\cos(dx)-1}{dx} + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx} \\
&= \sin(x)\lim_{dx\to 0}\left[\frac{\cos(dx)-1}{dx}\frac{\cos(dx)+1}{\cos(dx)+1}\right] + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx} \\
&= \sin(x)\lim_{dx\to 0}\frac{\cos^2(dx)-\cos(dx)+\cos(dx)-1}{dx(\cos(dx)+1)} + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx} \\
&= \sin(x)\lim_{dx\to 0}\frac{\cos^2(dx)-1}{dx(\cos(dx)+1)} + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx}
\end{aligned}$$

– Using (2.5) we can write:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \sin(x)\lim_{dx\to 0}\frac{-\sin^2(dx)}{dx(\cos(dx)+1)} + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx} \\
&= \sin(x)\lim_{dx\to 0}[-\sin(dx)]\cdot\lim_{dx\to 0}\frac{\sin(dx)}{dx}\cdot\lim_{dx\to 0}\frac{1}{\cos(dx)+1} + \cos(x)\lim_{dx\to 0}\frac{\sin(dx)}{dx}
\end{aligned}$$

– Since sin(dx)/dx=1 when dx→0 we can write:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \sin(x)\cdot 0\cdot 1\cdot\frac{1}{1+1} + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}$$

4.2. Proof

– We shall now prove following relation:

$$\cos'(x) = -\sin(x) \tag{2.39}$$

– We start with [1]:

$$\cos(x) = \sin(x+\frac{\pi}{2}) \bigg/ \frac{d}{dx}$$

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin(x+\frac{\pi}{2})$$

– By introducting:

$$u = x+\frac{\pi}{2} \tag{2.40}$$

we can write:

$$\frac{d}{dx}\cos(x) = \frac{d}{du}\sin(u)\cdot\frac{d}{dx}\left(x+\frac{\pi}{2}\right)$$

– Usgin (2.3) we can write:

$$\frac{d}{dx}\cos(x) = \cos(u)\cdot 1$$

– Inserting (2.38) back into the equation we get:

$$\frac{d}{dx}\cos(x) = \cos\left(x+\frac{\pi}{2}\right)$$

$$= -\sin(x)$$

4.3. Proof

– In this chapter we will prove equation (2.21):

$$\sum_{t=0}^{N-1} \cos^2(k\omega t) = \begin{cases} N & \text{for } k = 0 \\ \frac{N}{2} & \text{for } k \neq 0 \end{cases} \tag{2.41}$$

– Inserting (2.15) into (2.39) we get:

$$\sum_{t=0}^{N-1} \cos^2(k\omega t) = \sum_{t=0}^{N-1} \frac{1+\cos(2k\omega t)}{2} = \frac{1}{2}\sum_{t=0}^{N-1} 1 + \frac{1}{2}\sum_{t=0}^{N-1} \cos(2k\omega t) = \frac{N}{2} + \frac{1}{2}\sum_{t=0}^{N-1} \cos(2k\omega t)$$

– For k =0 we have:

$$\sum_{t=0}^{N-1} \cos^2(k\omega t) = \frac{N}{2} + \frac{1}{2}\sum_{t=0}^{N-1} 1 = \frac{N}{2} + \frac{1}{2} N = N$$

– For k ≠0, because of (3.4), second sum is equal to zero so we have:

$$\sum_{t=0}^{N-1} \cos^2(k\omega t) = \frac{N}{2}$$

4.4. Proof

– In this chapter we will prove equation (2.22):

$$\sum_{t=0}^{N-1} \sin^2(k\omega t) = \begin{cases} 0 & \text{for } k = 0 \\ \frac{N}{2} & \text{for } k \neq 0 \end{cases} \tag{2.42}$$

– Inserting (2.15) into (2.40) we get:

$$\sum_{t=0}^{N-1} \sin^2(k\omega t) = \sum_{t=0}^{N-1} \frac{1-\cos(2k\omega t)}{2} = \frac{1}{2}\sum_{t=0}^{N-1} 1 - \frac{1}{2}\sum_{t=0}^{N-1} \cos(2k\omega t) = \frac{N}{2} - \frac{1}{2}\sum_{t=0}^{N-1} \cos(2k\omega t)$$

– For k =0 we have:

$$\sum_{t=0}^{N-1} \sin^2(k\omega t) = \frac{N}{2} - \frac{1}{2}\sum_{t=0}^{N-1} 1 = \frac{N}{2} - \frac{1}{2} N = 0$$

– For k ≠0, because of (3.5), second sum is equal to zero so we have:

$$\sum_{t=0}^{N-1} \sin^2(k\omega t) = \frac{N}{2}$$

4.5. Proof

– In this chapter we will prove equation (2.17):

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \begin{cases} 2\pi & \text{for } \varphi = 0 \\ \pi & \text{for } \varphi \neq 0 \end{cases} \quad (2.43)$$

– Inserting (2.15) into (2.17) we get:

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \int_0^{2\pi} \frac{1+\cos(2\varphi)}{2}d\varphi = \frac{1}{2}\int_0^{2\pi} d\varphi + \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi = \frac{1}{2}2\pi + \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi = \pi + \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi$$

– For φ =0 we have:

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \pi + \frac{1}{2}\int_0^{2\pi} d\varphi = \pi + \frac{1}{2}2\pi = 2\pi$$

– For φ ≠0, because of 0 (2.43) second integral is equal to zero so we have:

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \pi$$

4.6. Proof

– In this chapter we will prove equation :

$$\int_0^{2\pi} \sin^2(\varphi)d\varphi = \begin{cases} 0 & \text{for } \varphi = 0 \\ \pi & \text{for } \varphi \neq 0 \end{cases} \quad (2.44)$$

– Inserting (2.16) into (2.42) we get:

$$\int_0^{2\pi} \sin^2(\varphi)d\varphi = \int_0^{2\pi} \frac{1-\cos(2\varphi)}{2}d\varphi = \frac{1}{2}\int_0^{2\pi} d\varphi - \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi = \frac{1}{2}2\pi - \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi = \pi - \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi$$

$$\int_0^{2\pi} \sin^2(\varphi)d\varphi = \pi - \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)d\varphi$$

– For φ =0 we have:

$$\int_0^{2\pi} \sin^2(\varphi)d\varphi = \pi - \frac{1}{2}\int_0^{2\pi} d\varphi = \pi - \frac{1}{2}2\pi = \pi - \pi = 0$$

– For φ ≠0, because of 0 (2.43) second integral is equal to zero so we have:

$$\int_0^{2\pi} \sin^2(\varphi)d\varphi = \pi - \frac{1}{2}\cdot 0 = \pi$$

4.7. Proof

$$\int_{-\pi}^{\pi} \cos(K\varphi)d(\varphi) = \frac{\sin(K\varphi)}{K} \Big/_{-\pi}^{\pi} \quad , \quad K \neq 0 \tag{2.45}$$

$$= \frac{1}{K}\left(\sin(K\pi) - \sin(-K\pi)\right)$$

$$= \frac{1}{K}(0 - 0)$$

$$= 0$$

4.8. Proof

$$\int_{-\pi}^{\pi} \sin(K\varphi)d(\varphi) = -\frac{\cos(K\varphi)}{K} \Big/_{-\pi}^{\pi} \quad , \quad K \neq 0 \tag{2.46}$$

$$= -\frac{1}{K}\left(\cos(K\pi) - \cos(-K\pi)\right)$$

$$= \frac{1}{K}\left(\cos(K\pi) - \cos(K\pi)\right)$$

$$= 0$$

5. References

[1] http://math2.org/math/derivatives/more/trig.htm