2D Descrete Fourier Sequence

1. Two Dimensional Discrete Fourier Sequence (2D-DFS)

    1.1. 2D-DFS The Big Picture

    1.2. 2D-DFS with ak and bk
        1.2.1. Calculating ak
        1.2.2. Calculating bk

    1.3. Connection between Ck, ak and bk coefficients

    1.4. Examples

    1.5. References

1. Two Dimensional Discrete Fourier Sequence (2D-DFS)

2D-DFS is a way of representing 2D descrete periodical signal with infinite sum of 2D descrete sinus and cosinus signals. 2D descrete signal is defined with folowing values:

t	–	discrete time index
f(t)	–	signal value at descrete time step t
N	–	number of descrete time steps in x and y direction
ω	–	frequency of the signal

Equations for 2D-DFS using A_k and φ_k are as folows:

$$f(x,y) = A(0,0) + \sum_{k=1}^{N/2}\sum_{\ell=1}^{N/2} A(k,\ell)\sin(k\omega x + \ell\omega y + \varphi_k) \tag{1.1}$$

Equations for 2D-DFS using a_k and b_k are as folows:

$$f(x,y) = a(0,0) + \sum_{k=1}^{N/2}\sum_{\ell=1}^{N/2}\left[a(k,\ell)\cos(k\omega x + \ell\omega y) + b(k,\ell)\sin(k\omega x + \ell\omega y)\right] \tag{1.2}$$

$$a(k,\ell) = \frac{2}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x + \ell\omega y) \quad , \qquad k=0,1,2,...,N/2 \tag{1.3}$$

$$b(k,\ell) = \frac{2}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x + L\omega y)\,, \qquad k=1,2,...,N/2 \tag{1.4}$$

Equations for 2D-DFS using C_k are as folows:

$$f(x,y) = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} C(k,\ell)e^{jk\omega x}e^{j\ell\omega y}$$

$$C(k,\ell) = \frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1} f(y,x)e^{-jk\omega x}e^{-j\ell\omega y} \quad , \qquad k=0,1,2,...,N-1 \tag{1.5}$$

1.1. 2D-DFS – The Big Picture

– In this chapter a brief owerview is given so that reader can have a big picture in mind while reading following chapters withouth getting lost in details.

– 2D-DFS is all about assuming that any 2D descrete periodical signal can be represented as sum of 2D descrete sinus signals in folowing way:

$$f(x,y) = A(0,0) + \sum_{k=1}^{N/2}\sum_{\ell=1}^{N/2} A(k,\ell)\sin\left[k\omega x + \ell\omega y + \varphi(k,\ell)\right] \tag{1.6}$$

– Extracting A(k,l) from (1.6) is easy but extracting φ(k,l) isn't.
– That is why we shall transform (1.6) into form in which unknowns are better seperated for easier extraction.
– Transformation starts by using (2.9) after which we get:

$$\begin{aligned} A(k,\ell)\sin\left[k\omega x + \ell\omega y) + \varphi(k,\ell)\right] &= A(k,\ell)\sin\left[k\omega x + \ell\omega y)\right]\cos\left[\varphi(k,\ell)\right] + A(k,\ell)\cos\left[k\omega x + \ell\omega y\right]\sin\left[\varphi(k,\ell)\right] = \\ &= A(k,\ell)\cos\left[\varphi(k,\ell)\right]\sin\left[k\omega x + \ell\omega y\right] + A(k,\ell)\sin\left[\varphi(k,\ell)\right]\cos\left[k\omega x + \ell\omega y\right] = \\ &= A(k,\ell)\sin\left[\varphi(k,\ell)\right]\cos\left[k\omega x + \ell\omega y)\right] + A(k,\ell)\cos\left[\varphi(k,\ell)\right]\sin\left[k\omega x + \ell\omega y\right] \end{aligned} \tag{1.9}$$

To continue we would need folowing substitutions:

$$a(0,0) = A(0,0) \tag{1.7}$$

$$a(k,\ell) = A(k,\ell)\sin\left[\varphi(k,\ell)\right] \tag{1.8}$$

$$b(k,\ell) = A(k,\ell)\cos\left[\varphi(k,\ell)\right] \tag{1.9}$$

Inserting (1.8) and (1.9) into (1.9) we get:

$$A(k,\ell)\sin\left[k\omega(x+y) + \varphi(k,\ell)\right] = a(k,\ell)\cos\left[k\omega x + \ell\omega y)\right] + b(k,\ell)\sin\left[k\omega x + \ell\omega y)\right] \tag{1.10}$$

Inserting (1.10) and (1.7) into (1.6) we get:

$$f(x,y) = a(0,0) + \sum_{k=1}^{N/2}\sum_{\ell=1}^{N/2} a(k,\ell)\cos\left[k\omega x + \ell\omega y)\right] + b(k,\ell)\sin\left[k\omega x + \ell\omega y)\right] \tag{1.11}$$

Extracting a(k,l) and b(k,l) from (1.11) is easy. To do so we shall use folowing property of sinus and cosinus signals. If you multiplie sinus signal with another sinus signal of frequency k times greater or smaller, and sum the result over on period, you will get zero. You will get zero even of you multiplie it with cosinus signal even if that signal has the same frequency. Only if you multiplie it by using sinus signal of the same frequency you will get result different then zero. Similar goes for cosinus signal.

So, to calculate a(k,l) for k≠0 and l≠0 we shall multiplie signal f(t) with cos(kωx+ lωy) and calculate sum over one period:

$$\begin{aligned} \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[k\omega x + \ell\omega y)\right] &= \sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\left[\frac{a(0,0)}{2} + a(k,\ell)\cos\left[k\omega x + \ell\omega y)\right]\right]\cos\left[k\omega x + \ell\omega y)\right] = \\ &= \frac{a(0,0)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left(k\omega x + \ell\omega y\right) + a(k,\ell)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos^2\left(k\omega x + \ell\omega y\right) \end{aligned} \tag{1.12}$$

For k≠0 and l≠0, first double sum is equal to zero because of ???? and second one is equal to ???? because of ???? so we get:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[k\omega x + \ell\omega y)\right] = a(k,\ell)\frac{N^2}{2} \tag{1.13}$$

So, to calculate a(0,0) we shall simple calculate sum of f(t) over one period:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} \frac{a(0,0)}{2} = \frac{a(0,0)}{2} N^2 \tag{1.14}$$

Now it is clear why we used a(0,0)/2 in (1.14) instead just a(0,0) because now we can use the same formula (1.3) to calculate a(k,l) both for (k=0,l=0) and (k≠0,l≠0).
In the similar way we get that b(k,l) can be calculated with (1.4) with difference that multiplication is done using sin(kωx+lωt) and b(0,0) is not calculated because it has no meaning.
Now that we have calculated a(k,l) and b(k,l) we can use (1.3) and (1.4) to calculate A(k,l) and φ(k,l).
Amplitude A(k,l) can be calculated as folows:

$$a^2(k,\ell) + b^2(k,\ell) = A^2(k,\ell)\left[\sin^2[\varphi(k,\ell)] + \cos^2[\varphi(k,\ell)]\right]$$

$$a^2(k,\ell) + b^2(k,\ell) = A^2(k,\ell)$$

$$A(k,\ell) = \sqrt{a^2(k,\ell) + b^2(k,\ell)}$$

Phase φ_k can be calculated as folows:

$$\frac{a(k,\ell)}{b(k,\ell)} = \frac{A(k,\ell)\sin[\varphi(k,\ell)]}{A(k,\ell)\cos[\varphi(k,\ell)]} = \tan[\varphi(k,\ell)]$$

$$\varphi(k,\ell) = \operatorname{cotan}\left(\frac{a(k,\ell)}{b(k,\ell)}\right) \tag{1.15}$$

If we create complex number or vector using a_k and b_k like this:

$$K(k,\ell) = a(k,\ell) + jb(k,\ell) \tag{1.16}$$

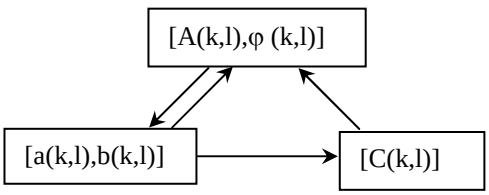
we see that its amplitude and phase are calculated in the same way as paramters A(k,l) and φ(k,l). Is this just coincidence or is there some hidden reason for it remains to be seen. This means that we can use K(k,l) to package coefficients a(k,l) and b(k,l). The problem with K(k,l) is that it can not be used to create formula similar to (1.1) and (1.2). For this reason a different complex parameter is introduced and it is defined as folows:

$$C(k,\ell) = \frac{1}{2}\left[a(k,\ell) - jb(k,\ell)\right] \tag{1.17}$$

Now we can write:

$$C(k,\ell) = \frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1} f(y,x) e^{-jk\omega x} e^{-j\ell\omega y} \tag{1.18}$$

The whole procedure could be sumorized as folows. To calculate A(k,l) and φ(k,l) first we calculate a(k,l) and b(k,l). From them we construct C(k,l) which we use to calculate A(k,l) and φ(k,l). This can be presented with folowing triangle:



Parameters a(k,l) and b(k,l) are calculated by scaning the signal. Each scaned is done by multipling the signal with signal of kω frequency and integrating the result over one period which gives us information of the amplitude of the signal we were scaning for.

1.2. 2D-DFS with a_k and b_k

Let us suppose that any two dimensional periodical function with frequency ω can be presented as the sum of 2D cosinus and sinus periodical functions, like shown below:

$$f(x,y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\right] \tag{1.19}$$

1.2.1. Calculating a_k

To calculate some concrete coeffcient a_K we shall multiplie equation (1.2) as folows:

$$f(x,y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\right]\Big/\cos(K\omega x+L\omega y) \tag{1.20}$$

$$f(x,y)\cos(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)\cos(K\omega x+L\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\cos(K\omega x+L\omega y)\right] \tag{1.21}$$

Using (2.4) and (2.5) we can transform this to:

$$\begin{aligned}f(x,y)\cos(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&a(k,\ell)\frac{\cos\left[(k\omega x+\ell\omega y)+(K\omega x+L\omega y)\right]+\cos\left[(k\omega x+\ell\omega y)-(K\omega x+L\omega y)\right]}{2}\\&+b(k,\ell)\frac{\sin\left[(k\omega x+\ell\omega y)+(K\omega x+L\omega y)\right]+\sin\left[(k\omega x+\ell\omega y)-(K\omega x+L\omega y)\right]}{2}\Bigg]\end{aligned} \tag{1.22}$$

$$\begin{aligned}f(x,y)\cos(K\omega x+K\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y\right]+\frac{b(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\Bigg]\end{aligned} \tag{1.23}$$

We now have isolated sin and cos. We can now use (2.16) and (2.16). If we integrate (1.23) we'll get:

$$\begin{aligned}f(x,y)\cos(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y\right]+\frac{b(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\Bigg]\Big/\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\end{aligned} \tag{1.24}$$

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos(K\omega x+L\omega y)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y\right]+\frac{b(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\Bigg]\end{aligned}$$

Inverting sumation order we get:

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\Bigg[&\frac{a(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y\right]+\frac{b(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\Bigg]\end{aligned}$$

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\frac{a(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k+K)\omega x+(\ell+L)\omega y\right]+\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\Bigg]\end{aligned}$$

Because of (2.16) and (2.16) all sumations, where (k+K) or (k-K) are different from zero, are equal to zero. This holds for all k≠K. This is definitly not the case where k=K so this is the only part which is not equal to zero.

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(K\omega x + L\omega y) = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k+K)\omega x + (\ell+L)\omega y\right] + \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k-K)\omega x + (\ell-L)\omega y\right] + \frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k+K)\omega x + (\ell+L)\omega y\right] + \frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k-K)\omega x + (\ell-L)\omega y\right] \tag{1.25}$$

Both of the remaining integrals that have sin are also zero. Last one because K-K=0, which means that sin is zero, and therefore the whole integral is zero. If K≠0 then the remaining integral of sin is zero for the same reason why all parts inside sum were zero. If K=0, this sin is zero, and therefore its integral also. So finaly we have:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(K\omega x + L\omega y) = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k+K)\omega x + (\ell+L)\omega y\right] + \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k-K)\omega x + (\ell-L)\omega y\right] \tag{1.26}$$

Second integral in (1.26) is always the same because:

$$\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k-K)\omega x + (\ell-L)\omega y\right] = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos(0) = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}1 = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}N = a(k,\ell)\frac{N^2}{2} \tag{1.27}$$

Inserting (1.27) into (1.26) we get:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(K\omega x + L\omega y) = \frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k+K)\omega x + (\ell+L)\omega y\right] + a(k,\ell)\frac{N^2}{2} \tag{1.28}$$

To make formula (1.28) more general, K should be replaced with k. First integral in (1.28) is a bit problematic, because for k=0 it becomes (1.27), and for k≠0 is zero because of (2.16). That means that we have to continue to solve this equation in two different ways. For k>0 and for k=0.

For k>0 we have:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x + \ell\omega y) = a(k,\ell)\frac{N^2}{2} \tag{1.29}$$

$$a(k,\ell) = \frac{2}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x + \ell\omega y) \tag{1.30}$$

For k=0 we have:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x + \ell\omega y) = a(k,\ell)\frac{N^2}{2} + a(k,\ell)\frac{N^2}{2} = N^2 a(k,\ell) \tag{1.31}$$

$$a(k,\ell) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x + \ell\omega y) \tag{1.32}$$

$$a(0,0) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) \tag{1.33}$$

1.2.2. Calculating b_k

To calculate some concrete coeffcient b_K we shall multiplie equation (1.19) as folows:

$$f(x,y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\right]\Big/\sin(K\omega x+L\omega y) \tag{1.34}$$

$$f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\left[a(k,\ell)\sin(k\omega x+\ell\omega y)\cos(K\omega x+L\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\sin(K\omega x+L\omega y)\right] \tag{1.35}$$

Using (2.4) and (2.5) we can transform this to:

$$\begin{aligned}f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&a(k,\ell)\frac{\sin\left[(k\omega x+\ell\omega y)+(K\omega x+L\omega y)\right]+\sin\left[(k\omega x+\ell\omega y)-(K\omega x+L\omega y)\right]}{2}\\&+b(k,\ell)\frac{\cos\left[(k\omega x+\ell\omega y)-(K\omega x+L\omega y)\right]-\cos\left[(k\omega x+\ell\omega y)+(K\omega x+L\omega y)\right]}{2}\Bigg]\end{aligned} \tag{1.36}$$

$$\begin{aligned}f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]-\frac{b(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y\right]\Bigg]\end{aligned} \tag{1.37}$$

We now have isolated sin and cos. We can now use (2.16) and (2.17). If we integrate (1.37) we'll get:

$$\begin{aligned}f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]-\frac{b(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y\right]\Bigg]\Big/\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\end{aligned} \tag{1.38}$$

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\sin(K\omega x+L\omega y)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]-\frac{b(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y\right]\Bigg]\end{aligned} \tag{1.39}$$

Inverting sumation order we get:

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\Bigg[&\frac{a(k,\ell)}{2}\sin\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]-\frac{b(k,\ell)}{2}\cos\left[(k+K)\omega x+(\ell+L)\omega y\right]\Bigg]\end{aligned} \tag{1.40}$$

$$\begin{aligned}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\sin(K\omega x+L\omega y)=\sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\Bigg[&\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k+K)\omega x+(\ell+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(k-K)\omega x+(\ell-L)\omega y\right]\\&+\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k-K)\omega x+(\ell-L)\omega y\right]-\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(k+K)\omega x+(\ell+L)\omega y\right]\Bigg]\end{aligned}$$

Because of (2.16) and (2.17) all sumations, where (k+K) or (k-K) are different from zero, are equal to zero. This holds for all $k\neq K$. This is definitly not the case where k=K so this is the only part which is not equal to zero.

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x+L\omega y)=\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(K+K)\omega x+(L+L)\omega y)\right]+\frac{a(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sin\left[(K-K)\omega x+(L-L)\omega y\right]$$
$$\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K-K)\omega x+(L-L)\omega y\right]-\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K+K)\omega x+(L+L)\omega y\right]$$

Both of the remaining integrals that have sin are also zero. Last one because K-K=0, which means that sin is zero, and therefore the whole integral is zero. If K≠0 then the remaining integral of sin is zero for the same reason why all parts inside sum were zero. If K=0, this sin is zero, and therefore its integral also. So finaly we have:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x+L\omega y)=\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K-K)\omega x+(L-L)\omega y\right]-\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K+K)\omega x+(L+L)\omega y\right] \quad (1.41)$$

Second integral in (1.41) is always the same because:

$$\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K-K)\omega x+(L-L)\omega y\right]=\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos(0)=\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}1=\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}N=b(k,\ell)\frac{N^2}{2} \quad (1.42)$$

Inserting (1.42) into (1.41) we get:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x+L\omega y)=b(k,\ell)\frac{N^2}{2}-\frac{b(k,\ell)}{2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\cos\left[(K+K)\omega x+(L+L)\omega y\right] \quad (1.43)$$

To make formula (1.43) more general K should be replaced with k. First integral in (1.43) is a bit problematic, because for k=0 it becomes (1.42) which, and for k≠0 is zero because of (2.16). That means that we have to continue to solve this equation in two different ways. For k>0 and for k=0.
For k>0 we have:

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x+L\omega y)=b(k,\ell)\frac{N^2}{2} \quad (1.44)$$

$$b(k,\ell)=\frac{2}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\sin(K\omega x+L\omega y) \quad (1.45)$$

For k=0 we have b_0. From **Error! Not a valid link.** it is seen that b_0 multiplies sin(kωt) which is zero for k=0 and therefore it doesn't metter which b_0 we choose, since it is multpiling signal which is zero and which doesn't add up to final result. This is why b_0 is not calculated.

1.3. Connection between C_k, a_k and b_k coefficients

Using (2.1), equation (1.19) can be rewritten as:

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)e^{-jk\omega x}e^{-j\ell\omega y}$$

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\left[\cos(k\omega x)-j\sin(k\omega x)\right]\left[\cos(\ell\omega y)-j\sin(\ell\omega y)\right]$$

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\left\{\left[\cos(k\omega x)\cos(\ell\omega y)-\sin(k\omega x)\sin(\ell\omega y)\right]-j\left[\cos(k\omega x)\sin(\ell\omega y)+j\sin(k\omega x)\cos(\ell\omega y)\right]\right\} \quad (1.46)$$

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\left\{\left[\frac{\cos(k\omega x+\ell\omega y)+\cos(k\omega x-\ell\omega y)}{2}-\frac{\cos(k\omega x-\ell\omega y)-\cos(k\omega x+\ell\omega y)}{2}\right]-\right.$$
$$\left.-j\left[\frac{\sin(\ell\omega y+k\omega x)+\sin(\ell\omega y-k\omega x)}{2}+\frac{\sin(k\omega x+\ell\omega y)+\sin(k\omega x-\ell\omega y)}{2}\right]\right\}$$

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\left[\cos(k\omega x+\ell\omega y)-j\sin(k\omega x+\ell\omega y)\right]$$

$$C(k,\ell)=\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\cos(k\omega x+\ell\omega y)-j\frac{1}{N^2}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)\sin(k\omega x+\ell\omega y)$$

$$C(k,\ell)=\frac{a(k,\ell)}{2}-j\frac{b(k,\ell)}{2}$$

$$C(k,\ell)=\frac{1}{2}\left[a(k,\ell)-jb(k,\ell)\right]$$

1.4. Examples

- For a picture of dimensions NxN we have N/2 different frequences in each direction which combined gives $N^2/4$ different building blocks described with (1.6).
- For N=100 pixels, this means that we have 50 different frequences in each directions and total of 2500 different building blocks described with (1.6).
- Some of these building blocks are given on figure 1.1.

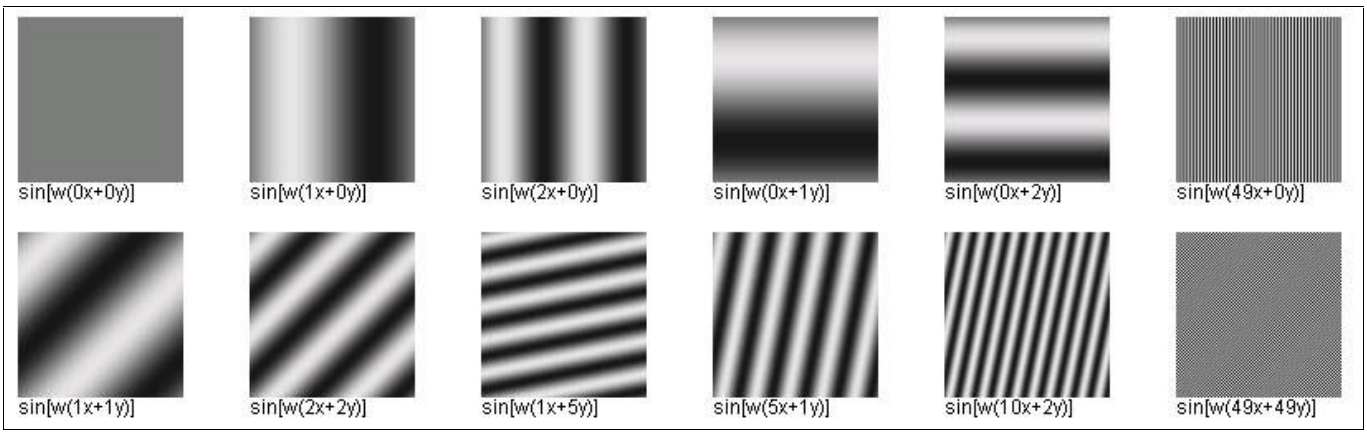


Fig.1.1. Building blocks of 2DDFT for picture 100x100 pixels.

- If we were to use frequences larger then N we would get the same building blocks as we did for lower frequnces.
- This is demonstrates on figure 1.2. for case where N=100.

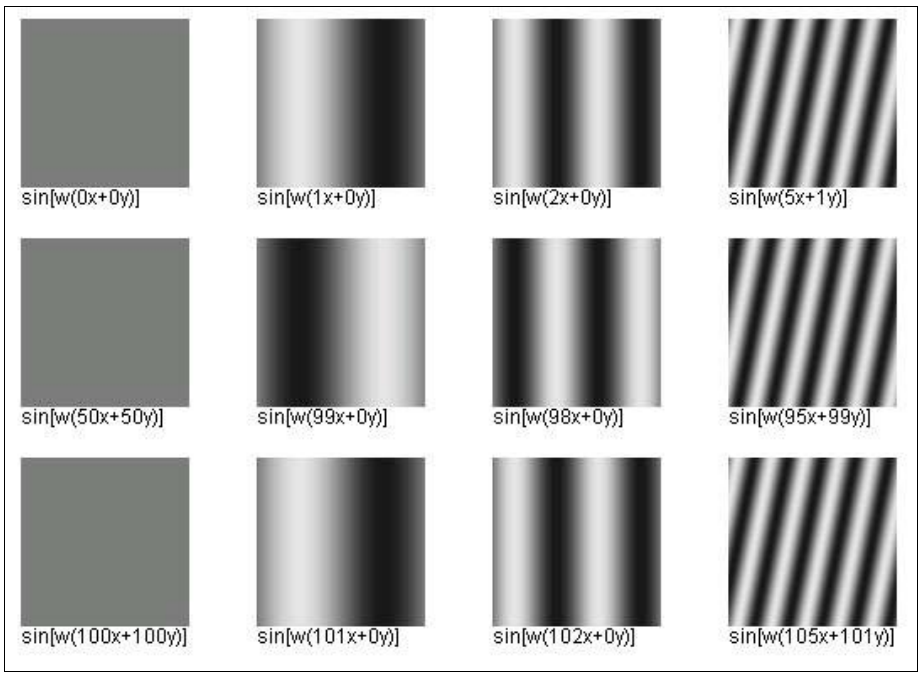


Fig.1.2. Extra frequences for pictire 100x100 pixels.

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos(k\omega x+\ell\omega y) = \sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\right]$$

$$\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) = \sum_{k=0}^{N/2}\sum_{\ell=0}^{N/2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\left[a(k,\ell)\cos(k\omega x+\ell\omega y)+b(k,\ell)\sin(k\omega x+\ell\omega y)\right]$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}. \qquad (1.47)$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}.$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}+\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}.$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

$$\sin\alpha\cos\beta-\sin\alpha\cos\beta=\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}-\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}.$$

$$\sum_{t=0}^{N-1}\cos(\omega t-\varphi)=\sum_{t=0}^{N-1}\cos(\omega t)\cos(\varphi)+\sin(\omega t)\sin(\varphi)=\cos(\varphi)\sum_{t=0}^{N-1}\cos(\omega t)+\sin(\varphi)\sum_{t=0}^{N-1}\sin(\omega t)=0$$

$$\sum_{t=0}^{N-1}\sin(\omega t-\varphi)=\sum_{t=0}^{N-1}\cos[\frac{\pi}{2}-(\omega t-\varphi)]=\sum_{t=0}^{N-1}\cos[-\omega t+\frac{\pi}{2}+\varphi]=\sum_{t=0}^{N-1}\cos[\omega t-(\frac{\pi}{2}+\varphi)]$$

$\sin\varphi=\cos(\pi/2-\varphi)$

$$\sin\alpha\cos\beta=\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}.$$

cossin+sincos

$$\sin\alpha\cos\beta+\sin\beta\cos\alpha=\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}+\frac{\sin(\beta+\alpha)+\sin(\beta-\alpha)}{2}=.\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}+\frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}=$$

Iznad nultog retka nacrtamo signal $C_k=\sin(k\omega x)$. Iznad svake kolone nacrtamo signal $C_\ell=\sin(\ell\omega y)$ kojemu je amplituda odredjan prethodno izracunatom signalu.
Uzmem signal $\sin(k\omega x)$.
S njim pomnozim svaki redak i odredim koliku amplitudu ima ta komponenta u svakom od redaka.

.48)

$$f(y,x)=\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}C(k,\ell)\sin(kwy)\sin(kwx)$$

Postavim se na nulti redak. Pixeli tog retka predstavljaju neku funkciju. Za tu funkciju izracunamo amplitudu signala $\sin(k\omega x)$.

$$C(r,c)=\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(y,x)e^{jc\omega x}e^{jr\omega y}$$

$$C(r)=\sum_{x=0}^{N-1}f(y,x)e^{jc\omega x}$$

Postavimo se na neki redak. Pixeli tog retka predstavljaju neku funckciju nad kojom onda napravimo DFT. Dobijemo vrijednost koeficijenta za taj redak C(r). Izracunamo te koeficijente za sve retke. Vertikalno gledajuci time smo dobili opet neku funkciju koeficiejanata nad kojima sada napravimo DFT.

$$C(r,c)=\sum_{y=0}^{N-1}C(r)e^{jr\omega y}$$

Pozicioniram se na prvi redak i za njega izracunam amplitudu signala $\sin(c\omega t)$.
Izracunam i za sve ostale retke.
Od hrpe tih koeficijenata napravim novu funkciju i za nju sada racunam amplitdu signala $\sin(r\omega t)$.
Taj konacni koeficijent predstavlja vrijednost DFT-a u (r,c).

$$f(n)=\sum_{k=0}^{N-1}C_k e^{jk\frac{2\pi}{N}n} \qquad (1.49)$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n) e^{-jk\frac{2\pi}{N}n} \tag{1.50}$$

$$f(n) = \sum_{k=0}^{N-1} C_k \left[\cos(k\frac{2\pi}{N}n) - j\sin(k\frac{2\pi}{N}n) \right]$$

$$f(n) = \sum_{k=0}^{N-1} \left[\frac{1}{N}\sum_{n=0}^{N-1} f(n)\cos(k\frac{2\pi}{N}n) - j\frac{1}{N}\sum_{n=0}^{N-1} f(n)\sin(k\frac{2\pi}{N}n) \right] \left[\cos(k\frac{2\pi}{N}n) - j\sin(k\frac{2\pi}{N}n) \right]$$

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} f(n)\cos(k\frac{2\pi}{N}n) - j\sum_{n=0}^{N-1} f(n)\sin(k\frac{2\pi}{N}n) \right] \left[\cos(k\frac{2\pi}{N}n) - j\sin(k\frac{2\pi}{N}n) \right]$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n) \left[\cos(k\frac{2\pi}{N}n) - j\sin(k\frac{2\pi}{N}n) \right]$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\cos(k\frac{2\pi}{N}n) - j\frac{1}{N}\sum_{n=0}^{N-1} f(n)\sin(k\frac{2\pi}{N}n)$$

1.5. References

2. Under Evaluation

In the above chapters we have used folowing trigonometric formulas:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \tag{2.1}$$

$$\sin(-\varphi) = -\sin(\varphi) \tag{2.2}$$

$$\cos(-\varphi) = \cos(\varphi) \tag{2.3}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}. \tag{2.4}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}. \tag{2.5}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}. \tag{2.6}$$

$$\int_{-\pi}^{\pi}\cos(K\varphi)d(\varphi) = \frac{\sin(K\varphi)}{K}\Big/_{-\pi}^{\pi} = \frac{1}{K}\left(\sin(K\pi)-\sin(-K\pi)\right) = \frac{1}{K}(0-0) = 0 \ , K\neq 0 \tag{2.7}$$

$$\int_{-\pi}^{\pi}\sin(K\varphi)d(\varphi) = -\frac{\cos(K\varphi)}{K}\Big/_{-\pi}^{\pi} = -\frac{1}{K}\left(\cos(K\pi)-\cos(-K\pi)\right) = \frac{1}{K}\left(\cos(K\pi)-\cos(K\pi)\right) = 0 \ , K\neq 0 \tag{2.8}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{2.9}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \tag{2.10}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{2.11}$$

$$\sum_{t=0}^{N-1}\cos^2(k\omega t) = \begin{cases} N & \text{for } k=0 \\ \frac{N}{2} & \text{for } k\neq 0 \end{cases} \tag{2.12}$$

$$\int_{0}^{2\pi}\cos^2(\varphi)d\varphi = \begin{cases} 2\pi & \text{for } \varphi=0 \\ \pi & \text{for } \varphi\neq 0 \end{cases} \tag{2.13}$$

Proof for (2.12):
Using (2.11) we can transform integral into folowing:

$$\sum_{t=0}^{N-1}\cos^2(k\omega t) = \sum_{t=0}^{N-1}\frac{1+\cos(2k\omega t)}{2} = \frac{1}{2}\sum_{t=0}^{N-1}1+\frac{1}{2}\sum_{t=0}^{N-1}\cos(2k\omega t) = \frac{N}{2}+\frac{1}{2}\sum_{t=0}^{N-1}\cos(2k\omega t)$$

For k =0 we have:

$$\sum_{t=0}^{N-1}\cos^2(k\omega t) = \frac{N}{2}+\frac{1}{2}\sum_{t=0}^{N-1}1 = \frac{N}{2}+\frac{1}{2}N = N$$

For k ≠0, because of (2.16) second integral is equal to zero so we have:

$$\sum_{t=0}^{N-1}\cos^2(k\omega t) = \frac{N}{2}$$

Proof for (2.13):
Using (2.11) we can transform integral into folowing:

$$\int_{0}^{2\pi}\cos^2(\varphi)d\varphi = \int_{0}^{2\pi}\frac{1+\cos(2\varphi)}{2}d\varphi = \frac{1}{2}\int_{0}^{2\pi}d\varphi+\frac{1}{2}\int_{0}^{2\pi}\cos(2\varphi)d\varphi = \frac{1}{2}2\pi+\frac{1}{2}\int_{0}^{2\pi}\cos(2\varphi)d\varphi = \pi+\frac{1}{2}\int_{0}^{2\pi}\cos(2\varphi)d\varphi$$

For φ =0 we have:

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \pi + \frac{1}{2}\int_0^{2\pi} d\varphi = \pi + \frac{1}{2}2\pi = 2\pi$$

For $\varphi \neq 0$, because of (2.8) second integral is equal to zero so we have:

$$\int_0^{2\pi} \cos^2(\varphi)d\varphi = \pi$$

2.1. Formulas used to obtain Fourier integral

$$\lim_{x \to 0} f(x) = f(dx) \tag{2.14}$$

$$\sum_{k=-\infty}^{\infty} f(kd\Omega)d\Omega = \int_{k=-\infty}^{\infty} f(\omega)d\omega \tag{2.15}$$

2.2. Formulas used for DFT

$$\sum_{t=0}^{N-1} \cos[(K\omega t] = 0 \tag{2.16}$$

$$\sum_{t=0}^{N-1} \sin[(K\omega t] = 0 \tag{2.17}$$

http://www.google.at/search?hl=en&ie=UTF-8&q=fourier+sequence