Continuous Fourier Sequence

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1. Overview

- Problem definition.

– Continuous Fourier Sequence (CFS) is a way of representing continuous periodical signal f(t):

f(t)	–	signal value at time t,
t	–	time index,
T	–	period of the signal,
ω	–	radial frequency of the signal,

where:

$$\omega = \frac{2\pi}{T} \tag{1.1}$$

with infinite sum of sine periodical signals which have:

- – different radial frequences $k\omega$, where k=0,1,2,....
- – different phase shifts φ_k.

- Different forms of CFS.

– Equations for Continuous Fourier Sequence (CFS) using A_k and φ_k:

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k) \tag{1.2}$$

– Equations for Continuous Fourier Sequence (CFS) using a_k and b_k:

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t) \right] \tag{1.3}$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)dt \qquad \text{(signal's mean value [2])} \tag{1.4}$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\cos(k\omega t)dt \quad , \ k=1,2,... \tag{1.5}$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\sin(k\omega t)dt \quad , \ k=1,2,... \tag{1.6}$$

– Equations for Continuous Fourier Sequence (CFS) using C_k are:

$$f(t) = C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ik\omega t} \tag{1.7}$$

$$C_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t)dt \qquad \text{(signal's mean value [2])} \tag{1.8}$$

$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-ik\omega t}dt \tag{1.9}$$

- Relations between coefficients of different forms.

– Calculating coefficients of the second form from coefficients of the main form:

$$a_0 = A_0 \tag{1.10}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.11}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.12}$$

– Calculating coefficients of the main form from coefficients of the second form:

$$A_0 = a_0 \tag{1.13}$$

$$A_k = \sqrt{a_k^2 + b_k^2} \quad , \quad k=1,2,... \tag{1.14}$$

$$\varphi_k = \text{arctan2}(a_k, b_k) \tag{1.15}$$

– Calculating coefficients of the complex form from coefficients of the second form:

$$C_0 = a_0 \tag{1.16}$$

$$C_k = \frac{1}{2}(a_k - i b_k) \quad , \quad k=1,2,... \tag{1.17}$$

– Calculating coefficients of the main form from coefficients of the complex form:

$$A_0 = C_0 \tag{1.18}$$

$$A_k = \frac{|C_k|}{2} \tag{1.19}$$

$$\varphi_k = \angle C_k + \frac{\pi}{2} \tag{1.20}$$

1.1. Transforming into second form

- In this chapter we will prove:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{1.21}$$

- Extracting A_k from (1.2) is easy, but extracting φ_k isn't:

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k) \tag{1.22}$$

– That is why we shall transform (1.22) into form in which unknowns are better separated for easier extraction.
– Using (5.4) [1] terms under sum in (1.22) can be transformed like this:

$$\begin{aligned} A_k \sin(k\omega t + \varphi_k) &= A_k \sin(k\omega t)\cos(\varphi_k) + A_k \cos(k\omega t)\sin(\varphi_k) \\ &= A_k \cos(\varphi_k)\sin(k\omega t) + A_k \sin(\varphi_k)\cos(k\omega t) \\ &= A_k \sin(\varphi_k)\cos(k\omega t) + A_k \cos(\varphi_k)\sin(k\omega t) \end{aligned} \tag{1.23}$$

– We continue by introducing following substitutions:

$$a_0 = A_0 \tag{1.24}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.25}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.26}$$

– Inserting (1.22) and (1.26) into (1.23), we get:

$$A_k \sin(k\omega t + \varphi_k) = a_k \cos(k\omega t) + b_k \sin(k\omega t) \tag{1.27}$$

– Inserting (1.24) and (1.27) into (1.22) we get (1.21):

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{1.28}$$

which we wanted to prove.

1.2. Transforming into complex form

- In this chapter we shall prove equation (1.7):

$$f(t) = C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ik\omega t} \tag{1.29}$$

- We start with equation (1.3):

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{1.30}$$

– We continue by using following relation which will be proven shortly:

$$\sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] = \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}(a_k - ib_k)\left[\cos(k\omega t) + i\sin(k\omega t)\right] \tag{1.31}$$

– Inserting (1.31) into (1.30) we get:

$$f(t) = a_0 + \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}(a_k - ib_k)\left[\cos(k\omega t) + i\sin(k\omega t)\right] \tag{1.32}$$

– Using following substitutions:

$$C_0 = a_0 \tag{1.33}$$

$$C_k = \frac{1}{2}(a_k - ib_k) \tag{1.34}$$

and Euler's formula (5.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{1.35}$$

equation (1.32) can be rewritten as:

$$f(t) = C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ik\omega t} \tag{1.36}$$

- And now to prove (1.31).

– By calculating the right side of (1.31) we get:

$$\sum_{k=1}^{\infty}\left[a_k\cos(k\omega t)+b_k\sin(k\omega t)\right] = \frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left\{\left[a_k\cos(k\omega t)+b_k\sin(k\omega t)\right]+i\left[a_k\sin(k\omega t)-b_k\cos(k\omega t)\right]\right\}$$

$$= \frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left[a_k\cos(k\omega t)+b_k\sin(k\omega t)\right]+i\frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left[a_k\sin(k\omega t)-b_k\cos(k\omega t)\right] \qquad (1.37)$$

– To calculate two sums in (1.37) following remarks have to be made.

– If we take one exact value for k, for instance k=-K, and put it into (1.5) and (1.6), using (5.2) and (5.3) [1] we get:

$$\begin{aligned} a_{-K} &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(-K\omega t)dt \\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt \\ &= a_K \end{aligned} \qquad (1.38)$$

$$\begin{aligned} b_{-K} &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(-K\omega t)dt \\ &= -\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt \\ &= -b_K \end{aligned} \qquad (1.39)$$

– From (1.38) and (1.39) and well known properties of sine and cosine, given by (5.2) and (5.3) [1], we can write:

$$\sin(-K\omega t) = -\sin(K\omega t)$$

$$\cos(-K\omega t) = \cos(K\omega t)$$

$$a_{-K} = a_K$$

$$b_{-K} = -b_K$$

– Using this four relations we will now calculate the result of adding two terms from first sum in (1.37) for k=K and k=-K:

$$\left[a_K\cos(K\omega t)+b_K\sin(K\omega t)\right]+\left[a_{-K}\cos(-K\omega t)+b_{-K}\sin(-K\omega t)\right] =$$

$$= \left[a_K\cos(K\omega t)+b_K\sin(K\omega t)\right]+\left[a_K\cos(K\omega t)+b_K\sin(K\omega t)\right] =$$

$$= 2\left[a_K\cos(K\omega t)+b_K\sin(K\omega t)\right] \qquad (1.40)$$

– Using the same method for the second sum in (1.37), we get:

$$\left[-a_K\sin(K\omega t)+b_K\cos(K\omega t)\right]+\left[-a_{-K}\sin(-K\omega t)+b_{-K}\cos(-K\omega t)\right] =$$

$$= \left[-a_K\sin(K\omega t)+b_K\cos(K\omega t)\right]+\left[a_K\sin(K\omega t)-b_K\cos(K\omega t)\right]$$

$$= 0 \qquad (1.41)$$

- From (1.40) we conclude that terms of first sum in (1.37) are symmetric around k=0 since terms with K and -K have the same values allowing us to rewrite fhat first sum like this:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[a_k \cos(k\omega t)+b_k \sin(k\omega t)\right] \quad = \quad 2\sum_{k=1}^{\infty}\left[a_k \cos(k\omega t)+b_k \sin(k\omega t)\right] \tag{1.42}$$

- From (1.41) we conclude that terms of second sum in (1.37) are also symetryc around k=0 but with different sign, since terms with K and -K have the same absolute values but different sign.
- This means that these terms are being canceled inside the sum making the whole sum equal to zero:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[a_k \sin(k\omega t)-b_k \cos(k\omega t)\right] \quad = \quad 0 \tag{1.43}$$

- Inserting (1.42) and (1.43) into (1.37) we get right side becomes equal to the left one ending our proof of (1.31).

1.3. Additional observations

- Reason for chosing time as integration variable

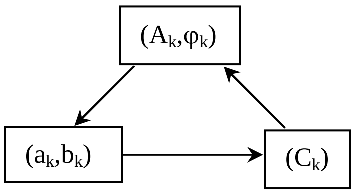
– There were three reasons for choosing integrals to go over time dt rather then over phase dωt that ids dφ.
– It is more intuitive to integrate over time since we usually think of signal as changing over time and that is the kind of image we keep in our minds.
– Also dωt seems a bit vague since it is easy to forget that this is acctually dφ and that ω stands for the frequency of original signal rather then frequency of some sine or cosine term.
– Last reason is the fact that Fourier and Laplace integrals are defined by integrating over time since they are used for signals that are not periodic and therefore T and ω don't exist for those signals.
– So using integration over time also for Fourier and Laplace sequences, where ω and T do exist, makes all these equations similar making it easier to understand how they are depended on each other.

- Reason for choosing integration bounders

– Integration bounders are chosen to go from -T/2 to T/2 rather then from 0 to T which would be simpler, becuase Fourier and Laplace integrals are defined to go from -∞ to ∞ and this is achieved by using -T/2 to T/2 and letting $T \to \infty$.

- Connecting coefficients

– The whole procedure could be summarized like this.
– To calculate A_0, A_k and φ_k first we calculate a_0, a_k and b_k.
– From them we construct C_k which we use to calculate A_k and φ_k.
– This can be presented with following triangle:



– Parameters A_0, A_k and φ_k are calculated by scaning the original signal f(t).
– Each scan is done by multiplying the signal with sine ar cosine of kω frequency and integrating the result over one period which gives us information of the amplitude and phase of the sine signal we were scanning for.

- Other forms of CFS

– There are also other versions of CFS then the ones described so far.

– For instance some rather use $a_0/2$ instead of a_0 in (1.3) so that a_0 can be calculated with the same equation as a_k:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{1.44}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k\omega t)d(\omega t) \quad , \ k=0,1,2,... \tag{1.45}$$

– Some define C_k differently then we have in (1.17) although the reason for this is not known to me:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t} \tag{1.46}$$

$$C_k = \frac{1}{2}\left(a_k - jb_k\right) \quad , \ k=0,1,2,... \tag{1.47}$$

2. Relation between coefficients

2.1. Relation between a_k and b_k and A_k and φ_k coefficients

- In this chapter we shall prove that relation between a_k and b_k and A_k and φ_k coefficients are like this:

$$A_0 = a_0 \tag{2.1}$$

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{2.2}$$

$$\varphi_k = \arctan\left(\frac{a_k}{b_k}\right) \tag{2.3}$$

- To prove the above equations we will use equations (1.24), (1.25) and (1.26) which defined coefficients a_k and b_k:

$$a_0 = A_0 \tag{2.4}$$

$$a_k = A_k \sin(\varphi_k) \tag{2.5}$$

$$b_k = A_k \cos(\varphi_k) \tag{2.6}$$

- Calculating A_0.

– Coefficient a_0 can be simply calculated from (2.4):

$$A_0 = a_0 \tag{2.7}$$

- Calculating amplitudes A_k.

– Amplitude A_k can be calculated by combining (2.5) and (2.6) like this:

$$a_k^2 + b_k^2 = A_k^2[\sin^2(\varphi) + \cos^2(\varphi)]$$

$$a_k^2 + b_k^2 = A_k^2$$

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{2.8}$$

- Calculating phase shifts φ_k.

– Phase shift φ_k can be calculated again by combining (2.5) and (2.6) like this:

$$\frac{a_k}{b_k} = \frac{A_k \sin(\varphi_k)}{A_k \cos(\varphi_k)}$$

$$= \tan(\varphi_k)$$

$$\varphi_k = \arctan\left(\frac{a_k}{b_k}\right) \tag{2.9}$$

2.2. Relation between C_k and A_k and φ_k coefficients

- In this chapter we shall prove that relation between C_k and A_k and φ_k coefficients is lilke this:

$$A_k = \frac{|C_k|}{2} \tag{2.10}$$

$$\varphi_k = \angle C_k + \frac{\pi}{2} \tag{2.11}$$

- Relation between $|C_k|$ and A_k.

– Coefficients C_k and A_k are defined with (1.14) and (1.17) like this:

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{2.12}$$

$$C_k = \frac{1}{2}(a_k - ib_k) \tag{2.13}$$

– From (1.17) we can write:

$$|C_k| = \frac{\sqrt{a_k^2 + b_k^2}}{2} \tag{2.14}$$

– Inserting (1.14) into **Error! Not a valid link.** we get:

$$|C_k| = \frac{A_k}{2} \tag{2.15}$$

- Relation between $\angle C_k$ and φ_k.

– Equation (1.15) defines φ_k like this:

$$\varphi_k = \arctan 2(a_k, b_k) \tag{2.16}$$

– Since equation (1.17) defines C_k like this:

$$C_k = \frac{1}{2}(a_k - ib_k) \quad , \quad k=1,2,... \tag{2.17}$$

angle of C_k, written as $\angle C$ is defined like this:

$$\angle C_k = \arctan 2(-b_k, a_k) \tag{2.18}$$

– Equations (1.15) and (1.13) are ilustrated on following figure:

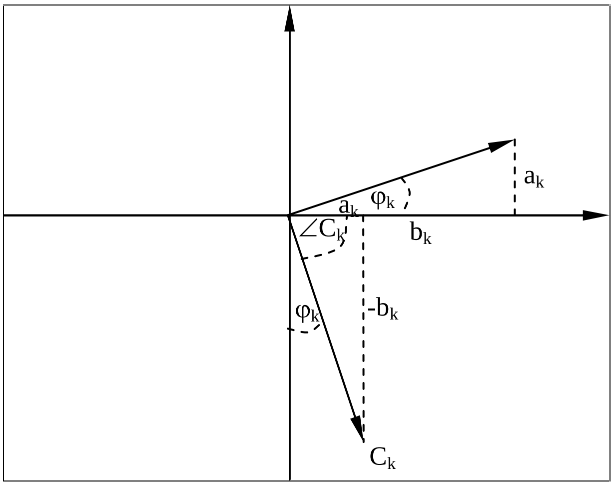


Figure 2.1. Relation betweenn φ_k and $\angle C_k$.

– From figure 2.1. we can see that relation between φ_k and $\angle C_k$ is:

$$\angle C_k = -\left(\frac{\pi}{2} - \varphi_k\right) \tag{2.19}$$

$$= \varphi_k - \frac{\pi}{2} \tag{2.20}$$

3. Calculating a_k and b_k

- In this chapter we will show how to find a_k and b_k from (1.3) proving that such appproximation is possible:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{3.1}$$

- How could we calculate some concrete coeficient a_K, where k=K?

– Idea is to multiply the equation (2.1) with something which would kill all terms with a_k and b_k, which are unknown in (2.1), but it wouldn't kill term with a_K, which we want to calculate.
– This way a_K will be left with things which are known and we would be able to calculate it.
– Each a_k and b_k should have something with which we could multiply (2.1) to calculate them as described above.

- To achieve this we shall use following properties of sine and cosine signals:

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{3.2}$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{3.3}$$

$$\int_{-T/2}^{T/2} \cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{3.4}$$

– If you multiply sine signal with another sine or cosine signal of frequency k times greater or smaller, or with the cosine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (4.13) only if you multiply sine signal with another sine signal of the same frequency you will get result different then zero.

– Same way if you multiply cosine signal with another sine or cosine signal of frequency k times greater or smaller, or with the sine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (4.14) only if you multiply cosine signal with another cosine signal of the same frequency you will get result different then zero.

- Calculating a_0:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \tag{3.5}$$

$$\int_{-T/2}^{T/2} f(t)dt = \int_{-T/2}^{T/2} dt a_0 = a_0 T \tag{3.6}$$

- Calculating a_k:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \cos(K\omega t) \Big/ \int_{-T/2}^{T/2} dt \tag{3.7}$$

$$\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt = \int_{-T/2}^{T/2} a_K \cos(K\omega t)\cos(K\omega t)dt = a_K \frac{T}{2} \quad , \text{ k=1,2,...} \tag{3.8}$$

- Calculating b_k:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \sin(K\omega t) \Big/ \int_{-T/2}^{T/2} dt \tag{3.9}$$

$$\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt = \int_{-T/2}^{T/2} b_K \sin(K\omega t)\sin(K\omega t)dt = b_K \frac{T}{2} \quad , \text{ k=1,2,...} \tag{3.10}$$

3.1. Calculating a_0

- To calculate a_0 we must take integral o equation (3.11) like this:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \qquad (3.11)$$

$$\int_{-T/2}^{T/2} f(t)dt = \int_{-T/2}^{T/2} a_0 dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k \cos(k\omega t)dt + \int_{-T/2}^{T/2} b_k \sin(k\omega t)dt\right] \qquad (3.12)$$

- Becuase of (5.11):

$$\int_{-T/2}^{T/2} \cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \qquad (3.13)$$

all terms defined with second integral on the right side are zero.

- Becuase of (5.12):

$$\int_{-T/2}^{T/2} \sin(k\omega t)dt = 0 \quad , \quad k \neq 0 \qquad (3.14)$$

all terms defined with second integral on the right side are zero.

- This leaves us with:

$$\int_{-T/2}^{T/2} f(t)dt = a_0 \int_{-T/2}^{T/2} dt$$

$$= a_0 1 \Big/_{-T/2}^{T/2}$$

$$= a_0\left(\frac{T}{2} - -\frac{T}{2}\right)$$

$$= a_0 T$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

3.2. Calculating a_k

- In this chapter we will prove:

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)dt \tag{3.15}$$

- To calculate some concrete coefficient a_K where K>0 we must multiply equation (3.11) like this:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t) + b_k\sin(k\omega t)\right] \Big/ \cos(K\omega t) \tag{3.16}$$

$$f(t)\cos(K\omega t) = a_0\cos(K\omega t) + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t)\cos(K\omega t) + b_k\sin(k\omega t)\cos(K\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \tag{3.17}$$

$$\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt = \int_{-T/2}^{T/2} a_0\cos(K\omega t)dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k\cos(k\omega t)\cos(K\omega t)dt + \int_{-T/2}^{T/2} b_k\sin(k\omega t)\cos(K\omega t)dt\right] \tag{3.18}$$

- Detecting terms equal to zero.

– Since K>0, becuase of (5.11):

$$\int_{-T/2}^{T/2}\cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{3.19}$$

first integral on the right side is zero.

– Becuse of (5.8):

$$\int_{-T/2}^{T/2}\sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{3.20}$$

all terms defined with last integral on the right side are also zero.

– Because of (5.10):

$$\int_{-T/2}^{T/2}\cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{3.21}$$

all terms defined with second integral on the right side are zero for k≠K leaving only one term when k=K:

$$\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt = \int_{-T/2}^{T/2} a_K\cos(K\omega t)\cos(K\omega t)dt \tag{3.22}$$

- Again because of (3.21) integral on the right side is T:

$$\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt = a_K\frac{T}{2} \tag{3.23}$$

$$a_K = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt \tag{3.24}$$

3.3. Calculating b_k

- In this chapter we will prove:

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)dt \tag{3.25}$$

- To calculate some concrete coefficient b_K where K>0 we must multiply equation (1.3) like this:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t) + b_k\sin(k\omega t)\right] \Big/ \sin(K\omega t) \tag{3.26}$$

$$f(t)\sin(K\omega t) = a_0\sin(K\omega t) + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t)\sin(K\omega t) + b_k\sin(k\omega t)\sin(K\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \tag{3.27}$$

$$\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt = \int_{-T/2}^{T/2} a_0\sin(K\omega t)dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k\cos(k\omega t)\sin(K\omega t)dt + \int_{-T/2}^{T/2} b_k\sin(k\omega t)\sin(K\omega t)dt\right] \tag{3.28}$$

- Detecting terms equal to zero.

– Since K>0, becuase of (5.12):

$$\int_{-T/2}^{T/2}\sin(k\omega)dt = 0 \quad , \quad k\neq 0 \tag{3.29}$$

first integral on the right side is zero.

– Becuse of (5.8):

$$\int_{-T/2}^{T/2}\sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{3.30}$$

all terms defined with secobd integral on the right side are also zero.

– Because of (5.10):

$$\int_{-T/2}^{T/2}\sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{3.31}$$

all terms defined with last integral on the right side are zero for k≠K which leaves only one term when k=K:

$$\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt = \int_{-T/2}^{T/2} b_K\sin(K\omega t)\sin(K\omega t)dt \tag{3.32}$$

- Again becuase of (3.31) integral on the right side is T:

$$\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt = b_K\frac{T}{2} \tag{3.33}$$

$$b_K = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt \tag{3.34}$$

4. Calculating C_k

– In this chapter we shall present two ways to caclulate C_k.

4.1. Calculating C_k from a_k and b_k

- In this chapter we shall prove (1.17):

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-ik\omega t}dt \tag{4.1}$$

- Coefficients C_k can be calculated from a_k and b_k using relation (1.17):

$$C_k = \frac{1}{2}\left(a_k - ib_k\right) \tag{4.2}$$

– Inserting (1.5) and (1.6):

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)dt \quad , \quad k=1,2,... \tag{4.3}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)dt \quad , \quad k=1,2,... \tag{4.4}$$

into (4.2) we get:

$$C_k = \frac{1}{2}\left[\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)dt - i\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\left[\cos\left(k\omega t\right) - i\sin\left(k\omega t\right)\right]dt \tag{4.5}$$

– Using Euler's formula (5.1) [1]:

$$e^{i\varphi} = \cos\left(\varphi\right) + i\sin\left(\varphi\right) \tag{4.6}$$

when $\varphi=-k\omega t$ we get:

$$e^{-ik\omega t} = \cos\left(-k\omega t\right) + i\sin\left(-k\omega t\right) \tag{4.7}$$

$$= \cos\left(k\omega t\right) - i\sin\left(k\omega t\right) \tag{4.8}$$

– Inserting (4.8) into (4.5) we get (4.1) which we wanted to prove.

4.2. Calculating C_k directly from complex definition

- In this chapter we shall prove (1.17):

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-ik\omega t}dt \qquad (4.9)$$

- One way to calculate C_k is directly from (1.7) like this:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega t} \quad \Big/ \quad e^{-iK\omega t} \qquad (4.10)$$

$$f(t)e^{-iK\omega t} = \sum_{k=-\infty}^{\infty} C_k e^{i(k-K)\omega t} \qquad (4.11)$$

$$= \sum_{k=-\infty}^{\infty} C_k e^{i(k-K)\omega t} \qquad (4.12)$$

– Using Euler's formula (5.1) [1] to rewritte part inside sum, and integrating the whole formula, results in:

$$f(t)e^{-iK\omega t} = \sum_{k=-\infty}^{\infty} C_k \left\{\cos[(k-K)\omega t] + j\sin[(k-K)\omega t]\right\} \Big/ \int_{-T/2}^{T/2} dt \qquad (4.13)$$

$$\int_{-T/2}^{T/2} f(t)e^{-iK\omega t}dt = \sum_{k=-\infty}^{\infty} C_k \left\{\int_{-T/2}^{T/2}\cos[(k-K)\omega t]dt + \int_{-T/2}^{T/2}\sin[(k-K)\omega t]dt\right\} \qquad (4.14)$$

– Because of (5.11) and (5.12):

$$\int_{-T/2}^{T/2}\cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \qquad (4.15)$$

$$\int_{-T/2}^{T/2}\sin(k\omega t)dt = 0 \quad , \quad k\neq 0 \qquad (4.16)$$

all integrals where (K+k),(K-k)≠0, are equal to zero, leaving integrals with k=K:

$$\int_{-T/2}^{T/2} f(t)e^{-iK\omega t}dt = C_K\int_{-T/2}^{T/2}\cos[(K-K)\omega t]dt + C_K\int_{-T/2}^{T/2}\sin[(K-K)\omega t]dt \qquad (4.17)$$

$$= C_K\int_{-T/2}^{T/2}\cos(0\omega t)dt + C_K\int_{-T/2}^{T/2}\sin(0\omega t)dt \qquad (4.18)$$

$$= C_K\int_{-T/2}^{T/2}1dt + C_K\int_{-T/2}^{T/2}0dt \qquad (4.19)$$

$$= C_K \Big/_{-T/2}^{T/2} + 0 \qquad (4.20)$$

$$= C_K\left(\frac{T}{2} - -\frac{T}{2}\right) \qquad (4.21)$$

$$= C_K T \qquad (4.22)$$

$$C_K = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-iK\omega t}dt \qquad (4.23)$$

– Formula (4.23) shows how to calculate concrete coefficient C_K so we can rewrite it by replacing K with k:
– By finding formula for C_k we have at the same time proven that it is possible to make approximation using (4.10).

5. Equations

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \quad (5.1)$$

$$\sin(-\varphi) = -\sin(\varphi) \quad (5.2)$$

$$\cos(-\varphi) = \cos(\varphi) \quad (5.3)$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad (5.4)$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \quad (5.5)$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \quad (5.6)$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \quad (5.7)$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \quad (5.8)$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \quad (5.9)$$

$$\int_{-T/2}^{T/2} \cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \quad (5.10)$$

$$\int_{-T/2}^{T/2} \cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \quad (5.11)$$

$$\int_{-T/2}^{T/2} \sin(k\omega t)dt = 0 \quad , \quad k \neq 0 \quad (5.12)$$

6. References

[1] Trigonometric Functions

[2] Mean Value.doc