CLI - Continuous Laplace Integral - Part I

1. Overview

– Equations with A_k and φ_k:

$$f(t) = e^{\sigma t}\int_0^{\infty}\frac{A_k}{d\omega}\sin(\tilde{\omega}t+\varphi_k)d\tilde{\omega} \tag{1.1}$$

– Equations with a_k and b_k:

$$f(t) = e^{\sigma t}\int_0^{\infty}\left[\frac{a_k}{d\omega}\cos(\tilde{\omega}t)+\frac{b_k}{d\omega}\sin(\tilde{\omega}t)\right]d\tilde{\omega} \tag{1.2}$$

$$\frac{a_k}{d\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty}e^{-\sigma t}f(t)\cos(\tilde{\omega}t)dt \tag{1.3}$$

$$\frac{b_k}{d\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin(\tilde{\omega}t)dt \tag{1.4}$$

– Equations with C_k:

$$f(t) = e^{\sigma t}\frac{1}{2}\int_{-\infty}^{\infty}\frac{C_k}{d\omega}e^{i\tilde{\omega}t}d\tilde{\omega} \tag{1.5}$$

$$\frac{C_k}{d\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty}e^{-\sigma t}f(t)e^{-i\tilde{\omega}t}dt \tag{1.6}$$

– More popular shape of C_k equations is like this:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(t)e^{st}d\tilde{\omega} \tag{1.7}$$

$$F(t) = \int_{-\infty}^{\infty}f(t)e^{-st}dt \tag{1.8}$$

$$s = \sigma + i\tilde{\omega} \tag{1.9}$$

2. Introduction

- Continuos Fourier Integral can't be calculated for functions that tend toward infinity for instance like function $f(t)=2^3$.
- But if we multiply this function with function $e^{-\sigma t}$ where $\sigma>0$ we will get function that tends towars zero.
- For such function we now can calculate CFI.
- So CLI is CFI of function f(t) which was multiplied with $e^{-\sigma t}$ so that it can have CFI.

- LCI is expressed using complex variable s which contains σ.
- Then LCI is analysied to determine for which σ LCI exists, which is called Reagion Of Convergence.

- Lets say that we have some system whose impuls response tends toward inifinity meaning that the system is unstable.
- Now lets say that we add some regulator ot that system such that impuls response is now $e^{-\sigma t}f(t)$.
- By taking CLI of such new signal we can se results of CLI to determine for which σ system will be stable.

- By doing CFI we want to get some info about f(t).
- Is this info lost if we do CFI of $e^{-\sigma t}f(t)$ instead?

- Laplace Transform of impulse response is called transfer function.
- Transfer function tells us about the stability of a system
- System is stable if and only if the transfer function has an RoC that includes the imaginary axis.
- This is the sam as saying that it is possible to find Fourier Transform of impulse response of the system since Fourier Transform is just a special case of Laplace Transform when $\sigma=0$.

- RoC of Laplace Transform of impulse response of a causal system

– Causal systems are those which react on input AFTER the input is created.
– This is like eating something and after that feeling full.
– All systems in our world are causal.

– So if we at time t=0 choose impuls function as input to our causal system then impuls response of causal system would be:

$$h(t) = 0 \quad \text{for } t < 0 \tag{2.1}$$

and Laplace Transform of such impulse response can be written like this:

$$H(t) = \int_0^{\infty} h(t)e^{-st}dt \tag{2.2}$$

where integration goes from 0 instead of $-\infty$

– The RoC of Laplace Transform of impulse response of a causal system always looks like this:

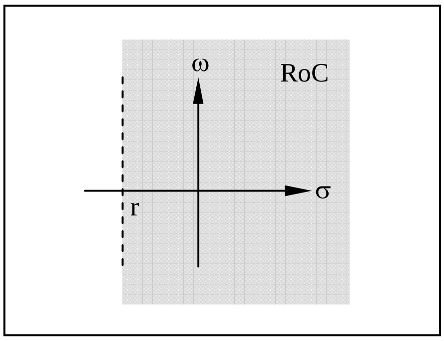

Figure 2.1. RoC of the transfer function of a causal system.

– From figure 2.1. it is seen that if RoC of Laplace Transform of impulse response of a causal system contains the line $\sigma = r$, it also contains the half-plane $\sigma > r$.
– This simply means that if $e^{-\sigma t}h(t)$ goes toward zero for some $\sigma=r$ it will also go toward zero for every $\sigma>r$ because $e^{-\sigma t}$ part will force expression $e^{-\sigma t}h(t)$ even faster toward 0.

- Finding RoC of Laplace Transform of impulse response of a causal system

– RoC can be very simply found by finding poles of transfer function.
– To do so transfer function needs to be written in following way:

$$H(t) = \frac{A(s)}{B(s)} \tag{2.3}$$

where: A,B – polynomials of s.

– The roots of the denominator B(s) of transfer function are called poles.
– The roots of the numerator A(s) of transfer function are called zeroes.
– RoC is right from the most right pole as shown on following figure:

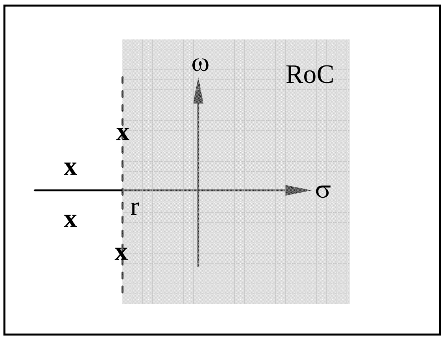

Figure 2.2. RoC of the transfer function of a causal system.

3. Proving equations

- In this chapter we shall prove equations which define CLI by starting with equations which define CFS:

– Equations with A_k and φ_k:

$$f(t) = e^{\sigma t}\left[A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)\right] \tag{2.4}$$

– Equations with a_k and b_k:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \tag{2.5}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt \tag{2.6}$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(k\omega t)dt \tag{2.7}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(k\omega t)dt \tag{2.8}$$

– Equations with C_k:

$$f(t) = e^{\sigma t}\left[C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ik\omega t}\right] \tag{2.9}$$

$$C_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt \tag{2.10}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)e^{-ik\omega t}dt \tag{2.11}$$

- For non periodic signal we can say that:

$$T \to \infty$$

and because of:

$$\omega = \frac{2\pi}{T}$$

this is the same as saying that:

$$\omega \to 0 \tag{2.12}$$

– Having in mind (2.12), for non-periodic signal, equations (2.4), (2.5) and (2.9) can be written using limes like this.

– Equations with A_k and φ_k:

$$e^{-\sigma t} f(t) = A_0 + \lim_{\omega \to 0} \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k) \tag{2.13}$$

– Equations with a_k and b_k:

$$e^{-\sigma t} f(t) = a_0 + \lim_{\omega \to 0} \sum_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t) \right] \tag{2.14}$$

$$a_0 = \lim_{\substack{\omega \to 0 \\ T \to \infty}} \frac{\omega}{2\pi} \int_{-T/2}^{T/2} e^{-\sigma t} f(t) dt \tag{2.15}$$

$$a_k = \lim_{\substack{\omega \to 0 \\ T \to \infty}} \frac{\omega}{\pi} \int_{-T/2}^{T/2} e^{-\sigma t} f(t) \cos(k\omega t) dt \tag{2.16}$$

$$b_k = \lim_{\substack{\omega \to 0 \\ T \to \infty}} \frac{\omega}{\pi} \int_{-T/2}^{T/2} e^{-\sigma t} f(t) \sin(k\omega t) dt \tag{2.17}$$

– Equations with C_k:

$$e^{-\sigma t} f(t) = C_0 + \frac{1}{2} \lim_{\omega \to 0} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} C_k e^{ik\omega t} \tag{2.18}$$

$$C_0 = \lim_{\substack{\omega \to 0 \\ T \to \infty}} \frac{\omega}{2\pi} \int_{-T/2}^{T/2} e^{-\sigma t} f(t) dt \tag{2.19}$$

$$C_k = \lim_{\substack{\omega \to 0 \\ T \to \infty}} \frac{\omega}{2\pi} \int_{-T/2}^{T/2} e^{-\sigma t} f(t) e^{-ik\omega t} dt \tag{2.20}$$

- We can get rid of limes simply by inserting dω instead of ω in the above equations.

– Equations with A_k and φ_k:

$$e^{-\sigma t} f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(kd\omega t + \varphi_k) \tag{2.21}$$

– Equations with a_k and b_k:

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(kd\omega t) + b_k \sin(kd\omega t)\right] \tag{2.22}$$

$$a_0 = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) dt \tag{2.23}$$

$$a_k = \frac{d\omega}{\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) \cos(kd\omega t) dt \tag{2.24}$$

$$b_k = \frac{d\omega}{\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) \sin(kd\omega t) dt \tag{2.25}$$

– Equations with C_k:

$$e^{-\sigma t} f(t) = C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ikd\omega t} \tag{2.26}$$

$$C_0 = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) dt \tag{2.27}$$

$$C_k = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) e^{-ikd\omega t} dt \tag{2.28}$$

- In the above equations $kd\omega$ can be observed as some new continuous variable $\tilde{\omega}$:

$$\tilde{\omega} = kd\omega \quad (2.29)$$

and from it's definition we get:

$$
\begin{aligned}
d\tilde{\omega} &= (k+1)d\omega - kd\omega \\
&= kd\omega + d\omega - kd\omega \\
&= d\omega
\end{aligned}
$$

– Using variable $\tilde{\omega}$, above equations can be transformed into integrals like this.

– Equations with A_k and φ_k:

$$e^{-\sigma t} f(t) = A_0 + \int_0^{\infty} \frac{A_k}{d\omega} \sin(\tilde{\omega} t + \varphi_k) d\tilde{\omega}$$

– Equations with a_k and b_k:

$$e^{-\sigma t} f(t) = a_0 + \int_0^{\infty} \left[\frac{a_k}{d\omega} \cos(\tilde{\omega} t) + \frac{b_k}{d\omega} \sin(\tilde{\omega} t) \right] d\tilde{\omega} \quad (2.30)$$

$$a_0 = 0 \quad (2.31)$$

$$\frac{a_k}{d\omega} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) \cos(\tilde{\omega} t) dt \quad (2.32)$$

$$\frac{b_k}{d\omega} = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\tilde{\omega} t) dt \quad (2.33)$$

– Equations with C_k:

$$e^{-\sigma t} f(t) = C_0 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{C_k}{d\omega} e^{j\tilde{\omega} t} d\tilde{\omega} \quad (2.34)$$

$$C_0 = 0 \quad (2.35)$$

$$\frac{C_k}{d\omega} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\sigma t} f(t) e^{-j\tilde{\omega} t} dt \quad (2.36)$$

– Since coefficient A_0, a_0 and C_0 are 0, equations can be written as presented at the begining of the tutorial which ends proof.

4. Equations

$$\lim_{x\to 0} f(x) = f(dx) \tag{2.37}$$

$$\sum_{k=-\infty}^{\infty} f(kd\Omega)d\Omega = \int_{-\infty}^{\infty} f(\omega)d\omega \tag{2.38}$$

$$\lim_{\Delta t\to 0} \sum_{k=1}^{T/\Delta t} f(k\Delta t) = \int_{t=0}^{T} \frac{f(t)}{dt}dt$$

5. References

[1] Trigonometric Functions

[2] Continuous Fourier Sequence (CFS)