CLS - Continuous Laplace Sequence

1. Overview

- Definition.

– Continuous Laplace Sequence (CLS) is a way of approximating continuous periodical signal f(t):

f(t)	–	signal value at time t,
t	–	time index,
T	–	period of the signal,
ω	–	radial frequency of the signal,

where:

$$\omega = \frac{2\pi}{T} \tag{1.1}$$

with infinite sum of sine periodical signals which have:

– common rate of growing or decaying over time defined through $e^{\sigma t}$,
– different radial frequences $k\omega$, where k=0,1,2,....
– different phase shifts φ_k.

- Differetn forms.

– Equations for Continuous Laplace Sequence (CLS) using A_k and φ_k:

$$f(t) = e^{\sigma t}\left[A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)\right] \tag{1.2}$$

– Equations for Continuous Laplace Sequence (CLS) using a_k and b_k:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \tag{1.3}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt \qquad \text{(signal's mean value [2])} \tag{1.4}$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(k\omega t)dt \quad , \quad k=1,2,3,... \tag{1.5}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(k\omega t)dt \quad , \quad k=1,2,3,... \tag{1.6}$$

– Equations for Continuous Fourier Sequence (CFS) using C_k are:

$$f(t) = e^{\sigma t}\left\{C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{-ik\omega t}\right\} \tag{1.7}$$

$$C_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt \qquad \text{(signal's mean value [2])} \tag{1.8}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)e^{ik\omega t}dt \quad , \quad k=1,2,3,... \tag{1.9}$$

- Relations between coefficients of different forms.

– Calculating coefficients of the second form from coefficients of the main form:

$$a_0 = A_0 \tag{1.10}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.11}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.12}$$

– Calculating coefficients of the main form from coefficients of the second form:

$$A_0 = a_0 \tag{1.13}$$

$$A_k = \sqrt{a_k^2 + b_k^2} \quad , \quad k=1,2,... \tag{1.14}$$

$$\varphi_k = \operatorname{cotan}\left(\frac{a_k}{b_k}\right) \tag{1.15}$$

– Calculating coefficients of the complex form from coefficients of the second form:

$$C_0 = a_0 \tag{1.16}$$

$$C_k = \frac{1}{2}\left(a_k - ib_k\right) \quad , \quad k=1,2,... \tag{1.17}$$

- Examples of building blocks.

– Figure 1.1. shows some examples of building blocks as defined with (1.2).
– Each of two columns on figure 1.1. show three building blocks and the function they approximate when combined.
– Figure 1.2. shows how A_0 influences the rate under which components increase or decrease.
– Figure 1.3. shows how σ influences the rate under which compoeenents increase or decrease.

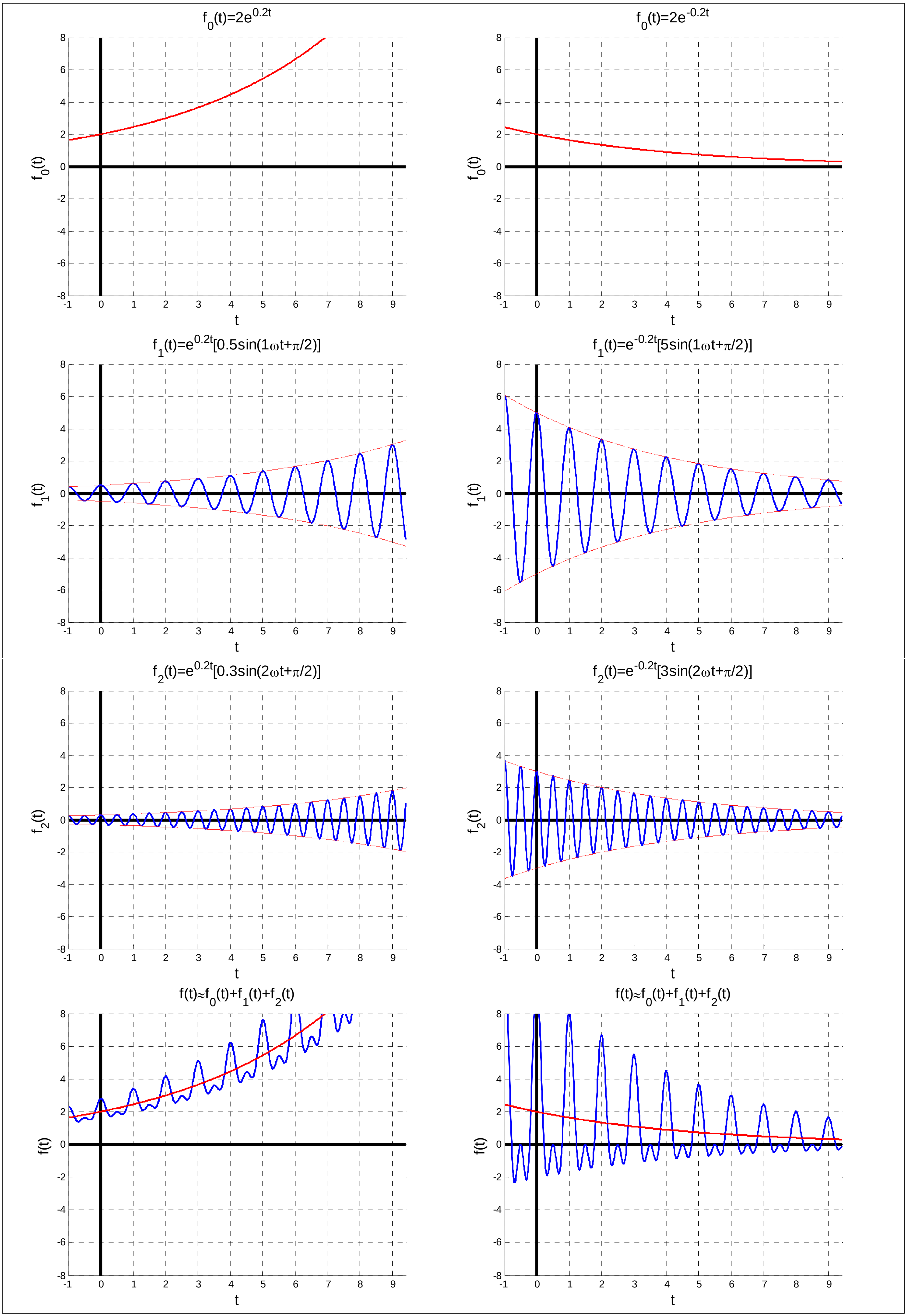


Figure 1.1. Examples of building blocks as defined with (1.2).

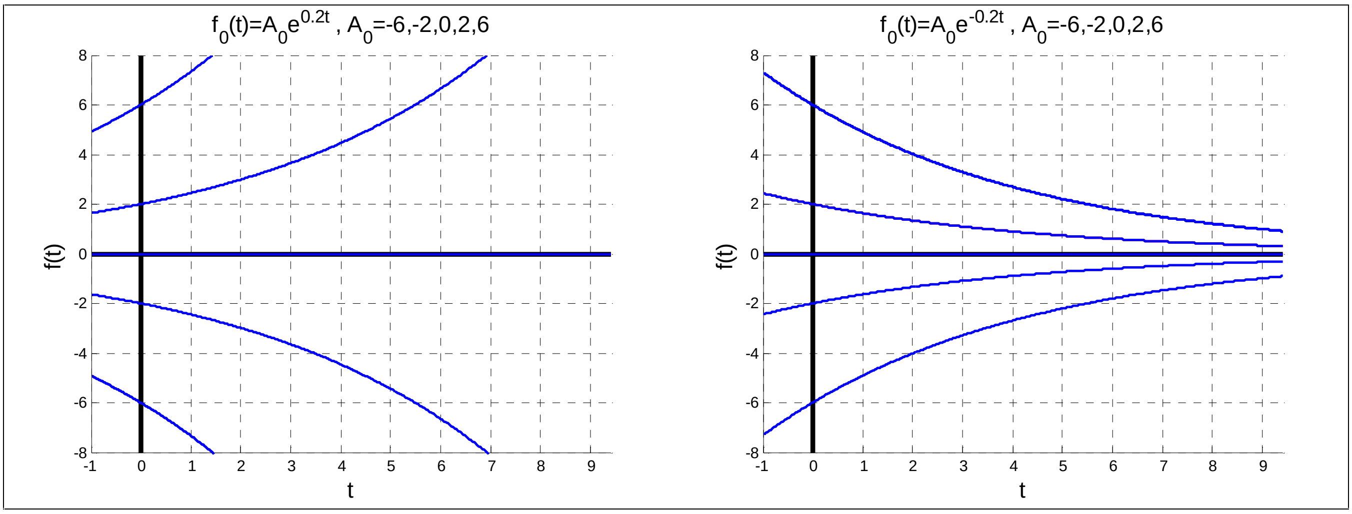


Figure 1.2. Influence of A_0.

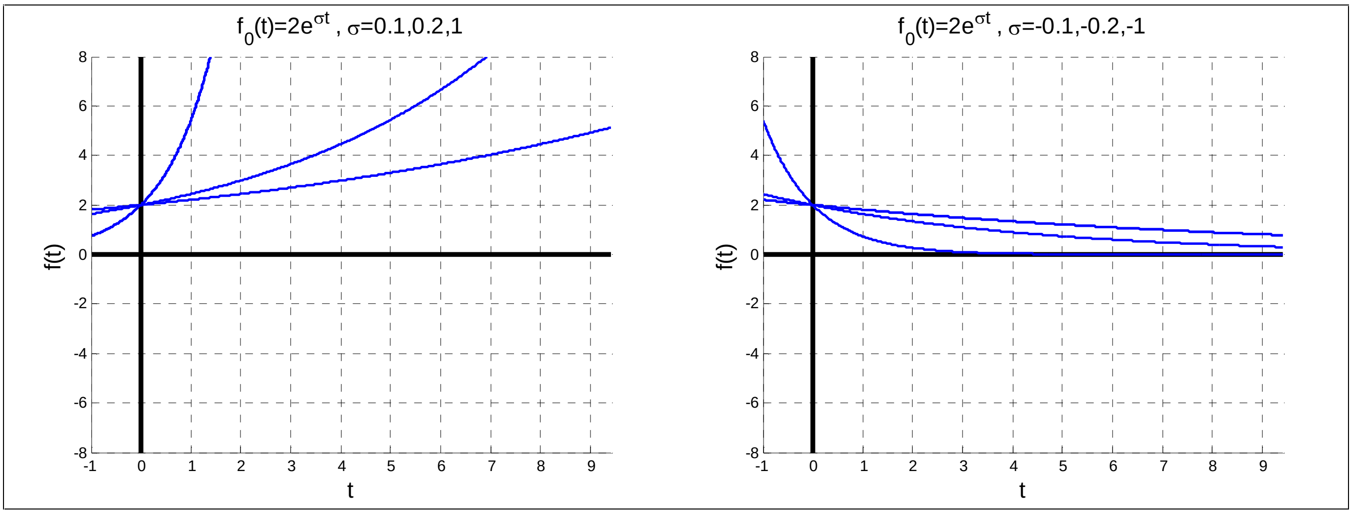


Figure 1.3. Influence of σ.

1.1. Transforming into second form

- In this chapter we will prove (1.3):

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \tag{1.18}$$

- Extracting A_k from (1.2) is easy, but extracting φ_k isn't:

$$f(t) = e^{\sigma t}\left[A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)\right] \tag{1.19}$$

– That is why we shall transform (1.19) into form in which unknowns are better separated for easier extraction.
– Using (4.4):

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{1.20}$$

terms under sum in (1.19) can be transformed like this:

$$\begin{aligned} A_k \sin(k\omega t + \varphi_k) &= A_k \sin(k\omega t)\cos(\varphi_k) + A_k \cos(k\omega t)\sin(\varphi_k) \\ &= A_k \cos(\varphi_k)\sin(k\omega t) + A_k \sin(\varphi_k)\cos(k\omega t) \\ &= A_k \sin(\varphi_k)\cos(k\omega t) + A_k \cos(\varphi_k)\sin(k\omega t) \end{aligned} \tag{1.21}$$

– We continue by introducing following substitutions:

$$a_0 = A_0 \tag{1.22}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.23}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.24}$$

– Inserting (1.23) and (1.24) into (1.21), we get:

$$A_k \sin(k\omega t + \varphi_k) = a_k \cos(k\omega t) + b_k \sin(k\omega t) \tag{1.25}$$

– Inserting (1.22) and (1.25) into (1.19) we get (1.18) which we wanted to prove.

- We will end this chapter by showing how to calculate A_0, A_k and φ_k from a_0, a_k and b_k.

– From (1.22) we can simply write:

$$A_0 = a_0 \tag{1.26}$$

– Amplitude A_k can be calculated by combining (1.23) and (1.24) like this:

$$a_k^2 + b_k^2 = A_k^2[\sin^2(\varphi) + \cos^2(\varphi)]$$

$$a_k^2 + b_k^2 = A_k^2$$

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{1.27}$$

– Phase shift φ_k can be calculated again by combining (1.23) and (1.24) like this:

$$\begin{aligned} \frac{a_k}{b_k} &= \frac{A_k \sin(\varphi_k)}{A_k \cos(\varphi_k)} \\ &= \tan(\varphi_k) \end{aligned}$$

$$\varphi_k = \operatorname{cotan}\left(\frac{a_k}{b_k}\right) \tag{1.28}$$

1.2. Transforming into complex form

- In this chapter we shall prove equation (1.7):

$$f(t) = e^{\sigma t}\left\{C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{-ik\omega t}\right\} \tag{1.29}$$

- We start with equation (1.3):

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \tag{1.30}$$

– We continue by using following relation which will be proven shortly:

$$\sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] = \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}(a_k - ib_k)\left[\cos(k\omega t) + i\sin(k\omega t)\right] \tag{1.31}$$

– Inserting (1.31) into (1.30) we get:

$$f(t) = e^{\sigma t}\left\{a_0 + \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}(a_k - ib_k)\left[\cos(k\omega t) + i\sin(k\omega t)\right]\right\} \tag{1.32}$$

– Using following substitutions:

$$C_0 = a_0 \tag{1.33}$$

$$C_k = \frac{1}{2}(a_k - ib_k) \tag{1.34}$$

and Euler's formula (4.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{1.35}$$

equation (1.32) can be rewritten as:

$$f(t) = e^{\sigma t}\left\{C_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_k e^{ik\omega t}\right\} \tag{1.36}$$

which is the same as (1.29) which we wanted to prove.

- And now to prove (1.31).

– By calculating the right side of (1.31) we get:

$$\sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] = \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left\{\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] + i\left[a_k \sin(k\omega t) - b_k \cos(k\omega t)\right]\right\}$$

$$= \frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] + i\frac{1}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[a_k \sin(k\omega t) - b_k \cos(k\omega t)\right] \quad (1.37)$$

– To calculate two sums in (1.37) following remarks have to be made.
– If we take one exact value for k, for instance k=-K, and put it into (1.5) and (1.6), using (4.2) and (4.3):

$$\sin(-\varphi) = -\sin(\varphi) \quad (1.38)$$

$$\cos(-\varphi) = \cos(\varphi) \quad (1.39)$$

we get:

$$a_{-K} = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(-K\omega t)dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(K\omega t)dt$$

$$= a_K \quad (1.40)$$

$$b_{-K} = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(-K\omega t)dt$$

$$= -\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(K\omega t)dt$$

$$= -b_K \quad (1.41)$$

– We will now rewrite equations (1.38), (1.39), (1.40) and (1.41) to have them netly at aone place:

$$\sin(-K\omega t) = -\sin(K\omega t)$$

$$\cos(-K\omega t) = \cos(K\omega t)$$

$$a_{-K} = a_K$$

$$b_{-K} = -b_K$$

– Using this four relations we will now calculate the result of adding two terms from first sum in (1.37) for k=K and k=-K:

$$\left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] + \left[a_{-K} \cos(-K\omega t) + b_{-K} \sin(-K\omega t)\right] =$$

$$= \left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] + \left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right]$$

$$= 2\left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] \quad (1.42)$$

– Using the same method for the second sum in (1.37), we get:

$$\left[-a_K \sin(K\omega t) + b_K \cos(K\omega t)\right] + \left[-a_{-K} \sin(-K\omega t) + b_{-K} \cos(-K\omega t)\right] =$$

$$= \left[-a_K \sin(K\omega t) + b_K \cos(K\omega t)\right] + \left[a_K \sin(K\omega t) - b_K \cos(K\omega t)\right]$$

$$= 0 \quad (1.43)$$

- From (1.42) we conclude that terms of first sum in (1.37) are symmetric around k=0 since terms with K and -K have the same values allowing us to rewrite fhat first sum like this:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[a_k \cos(k\omega t)+b_k \sin(k\omega t)\right] = 2\sum_{k=1}^{\infty}\left[a_k \cos(k\omega t)+b_k \sin(k\omega t)\right] \qquad (1.44)$$

- From (1.43) we conclude that terms of second sum in (1.37) are also symetryc around k=0 but with different sign, since terms with K and -K have the same absolute values but different sign.
- This means that these terms are being canceled inside the sum making the whole sum equal to zero:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left[-a_k \sin(k\omega t)+b_k \cos(k\omega t)\right] = 0 \qquad (1.45)$$

- Inserting (1.44) and (1.45) into (1.37) we get right side becomes equal to the left one ending our proof of (1.31).

1.3. Additional observations

- Reason for chosing time as integration variable

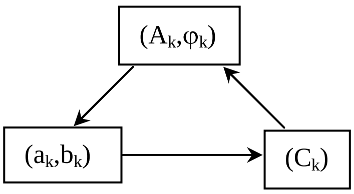
– There were three reasons for choosing integrals to go over time dt rather then over phase dωt that ids dφ.
– It is more intuitive to integrate over time since we usually think of signal as changing over time and that is the kind of image we keep in our minds.
– Also dωt seems a bit vague since it is easy to forget that this is acctually dφ and that ω stands for the frequency of original signal rather then frequency of some sine or cosine term.
– Last reason is the fact that Fourier and Laplace integrals are defined by integrating over time since they are used for signals that are not periodic and therefore T and ω don't exist for those signals.
– So using integration over time also for Fourier and Laplace sequences, where ω and T do exist, makes all these equations similar making it easier to understand how they are depended on each other.

- Reason for choosing integration bounders

– Integration bounders are chosen to go from -T/2 to T/2 rather then from 0 to T which would be simpler, becuase Fourier and Laplace integrals are defined to go from -∞ to ∞ and this is achieved by using -T/2 to T/2 and letting T→ ∞.

- Connecting coefficients

– The whole procedure could be summarized like this.
– To calculate A_0, A_k and φ_k first we calculate a_0, a_k and b_k.
– From them we construct C_k which we use to calculate A_k and φ_k.
– This can be presented with following triangle:



– Parameters A_0, A_k and φ_k are calculated by scaning the original signal f(t).
– Each scan is done by multiplying the signal with sine ar cosine of kω frequency and integrating the result over one period which gives us information of the amplitude and phase of the sine signal we were scanning for.

2. Calculating a_k and b_k

- In this chapter we will show how to calculate a_0, a_k and b_k from (1.2):

$$f(t) = e^{\sigma t}\left[A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)\right] \tag{2.1}$$

- How could we calculate some concrete coeficient a_K, where k=K?

– Idea is to multiply the equation (2.1) with something which would kill all a_k and b_k, which are unknowns in (2.1), but it wouldn't kill a_K, which we want to calculate.
– This way a_K will be left with things which are known and we would be able to calculate it.
– Each a_k and b_k should have its own thing with which we should multiply (2.1).

- To achieve this we shall use following properties of sine and cosine signals (4.8), (4.9) and (4.10):

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{2.2}$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.3}$$

$$\int_{-T/2}^{T/2} \cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{1}{T} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.4}$$

– If you multiply sine signal with another sine or cosine signal of frequency k times greater or smaller, or with the cosine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (4.9) only if you multiply sine signal with another sine signal of the same frequency you will get result different then zero.

– Same way if you multiply cosine signal with another sine or cosine signal of frequency k times greater or smaller, or with the sine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (4.10) only if you multiply cosine signal with another cosine signal of the same frequency you will get result different then zero.

- Calculating a_0:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \Big/ e^{-\sigma t} \Big/ \int_{-T/2}^{T/2} dt \tag{2.5}$$

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt = a_0 \int_{-T/2}^{T/2} dt = a_0 T$$

- Calculating a_k

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \Big/ e^{-\sigma t} \Big/ \cos(K\omega t) \Big/ \int_{-T/2}^{T/2} dt \tag{2.6}$$

$$\int_{0}^{2\pi} e^{-\sigma t} f(t)\cos(K\omega t)d(\omega t) = \int_{-T/2}^{T/2} a_K \cos(K\omega t)\cos(K\omega t)dt = a_K \frac{T}{2} \tag{2.7}$$

- Calculating b_k:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \Big/ e^{-\sigma t} \Big/ \sin(K\omega t) \Big/ \int_{-T/2}^{T/2} dt \tag{2.8}$$

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt = b_K \frac{T}{2} \tag{2.9}$$

2.1. Calculating a_0

- In this chapter we will prove (1.4):

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt \tag{2.10}$$

- To calculate a_0 we start with (1.3) like this:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \Big/ e^{-\sigma t} \tag{2.11}$$

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{2.12}$$

– The right side of the equation is exactly the same as in Fourier sequence, hence the remaining procedure is the same:

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \tag{2.13}$$

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt = \int_{-T/2}^{T/2} a_0 dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k \cos(k\omega t)dt + \int_{-T/2}^{T/2} b_k \sin(k\omega t)dt\right] \tag{2.14}$$

– Becuase of (4.11):

$$\int_{-T/2}^{T/2} \cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{2.15}$$

all terms defined with second integral on the right side are zero.

– Becuase of (4.12):

$$\int_{-T/2}^{T/2} \sin(k\omega t)dt = 0 \quad , \quad k \neq 0 \tag{2.16}$$

all terms defined with second integral on the right side are zero.

– This leaves us with:

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt = a_0 \int_{-T/2}^{T/2} dt$$

$$= a_0 1 \Big/_{-T/2}^{T/2}$$

$$= a_0\left(\frac{T}{2} - -\frac{T}{2}\right)$$

$$= a_0 T$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)dt$$

which is the same as (2.10) which we wanted to prove.

2.2. Calculating a_k

- In this chapter we will prove (1.5):

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(k\omega t)dt \quad , \quad k=1,2,3,... \tag{2.17}$$

- To calculate some concrete coefficient a_K where K>0 we must multiply equation (1.3) like this:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]\right\} \Big/ \ e^{-\sigma t} \tag{2.18}$$

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \tag{2.19}$$

– The right side of the equation is exactly the same as in Fourier sequence, hence the remaining procedure is the same:

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \Big/ \ \cos(K\omega t) \tag{2.20}$$

$$e^{-\sigma t} f(t)\cos(K\omega t) = a_0 \cos(K\omega t) + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega t)\cos(K\omega t) + b_k \sin(k\omega t)\cos(K\omega t)\right] \Big/ \int_{-T/2}^{T/2} dt \tag{2.21}$$

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(K\omega t)dt = \int_{-T/2}^{T/2} a_0 \cos(K\omega t)dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k \cos(k\omega t)\cos(K\omega t)dt + \int_{-T/2}^{T/2} b_k \sin(k\omega t)\cos(K\omega t)dt\right] \tag{2.22}$$

– Since K>0, becuase of (4.11):

$$\int_{-T/2}^{T/2} \cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{2.23}$$

first integral on the right side is zero.

– Becuse of (4.8):

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{2.24}$$

all terms defined with last integral on the right side are also zero.

– Because of (4.10):

$$\int_{-T/2}^{T/2} \cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.25}$$

all terms defined with second integral on the right side are zero for k≠K which leaves only one term when k=K:

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(K\omega t)dt = \int_{-T/2}^{T/2} a_K \cos(K\omega t)\cos(K\omega t)dt \tag{2.26}$$

– Again becuase of (2.25) integral on the right side is T/2:

$$\int_{0}^{2\pi} e^{-\sigma t} f(t)\cos(K\omega t)d(\omega t) = a_K \frac{T}{2} \tag{2.27}$$

$$a_K = \frac{1}{\pi}\int_{0}^{2\pi} e^{-\sigma t} f(t)\cos(K\omega t)d(\omega t) \tag{2.28}$$

which is the same as (2.17) which we wanted to prove.

2.3. Calculating b_k

- In this chapter we will prove (1.6):

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(k\omega t)dt \quad , \quad k=1,2,3,... \tag{2.29}$$

- To calculate some concrete coefficient b_K where K>0 we must multiply equation (1.3) like this:

$$f(t) = e^{\sigma t}\left\{a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t) + b_k\sin(k\omega t)\right]\right\} \Big/ \; e^{-\sigma t} \tag{2.30}$$

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t) + b_k\sin(k\omega t)\right] \tag{2.31}$$

– The right side of the equation is exactly the same as in Fourier sequence, hence the remaining procedure is the same:

$$e^{-\sigma t} f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t) + b_k\sin(k\omega t)\right] \Big/ \; \sin(K\omega t) \tag{2.32}$$

$$e^{-\sigma t} f(t)\sin(K\omega t) = a_0\sin(K\omega t) + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega t)\sin(K\omega t) + b_k\sin(k\omega t)\sin(K\omega t)\right] \Big/ \int_{-T/2}^{T/2} d(\omega t) \tag{2.33}$$

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt = \int_{-T/2}^{T/2} a_0\sin(K\omega t)dt + \sum_{k=1}^{\infty}\left[\int_{-T/2}^{T/2} a_k\cos(k\omega t)\sin(K\omega t)dt + \int_{-T/2}^{T/2} b_k\sin(k\omega t)\sin(K\omega t)dt\right] \tag{2.34}$$

– Since K>0, becuase of (4.12):

$$\int_{-T/2}^{T/2}\sin(k\omega)dt = 0 \quad , \quad k\neq 0 \tag{2.35}$$

first integral on the right side is zero.

– Becuse of (4.8):

$$\int_{-T/2}^{T/2}\sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \tag{2.36}$$

all terms defined with secobd integral on the right side are also zero.

– Because of (4.10):

$$\int_{-T/2}^{T/2}\sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.37}$$

all terms defined with last integral on the right side are zero for k≠K which leaves only one term when k=K:

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt = \int_{-T/2}^{T/2} b_K\sin(K\omega t)\sin(K\omega t)dt \tag{2.38}$$

– Again becuase of (2.37) integral on the right side is π:

$$\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt = b_K\frac{T}{2} \tag{2.39}$$

$$b_K = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(K\omega t)dt \tag{2.40}$$

which is the same as (2.29) which we wanted to prove.

3. Calculating C_k

– In this chapter we shall present two ways to caclulate C_k.

3.1. Calculating C_k from a_k and b_k

- In this chapter we shall prove (1.9):

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t) e^{ik\omega t} dt \quad , \quad k=1,2,3,... \tag{3.1}$$

- Coefficients C_k can be calculated from a_k and b_k using relation (1.17):

$$C_k = \frac{1}{2}\left(a_k - ib_k\right) \tag{3.2}$$

– Inserting (1.5) and (1.6):

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(k\omega t) dt \quad , \quad k=1,2,3,... \tag{3.3}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(k\omega t) dt \quad , \quad k=1,2,3,... \tag{3.4}$$

into (3.2) we get:

$$C_k = \frac{1}{2}\left[\frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\cos(k\omega t) dt - i\frac{2}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\sin(k\omega t) dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t)\left[\cos(k\omega t) - i\sin(k\omega t)\right] dt \tag{3.5}$$

– Using Euler's formula (4.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{3.6}$$

when $\varphi=-k\omega t$ we get:

$$e^{-ik\omega t} = \cos(-k\omega t) + i\sin(-k\omega t) \tag{3.7}$$

$$= \cos(k\omega t) - i\sin(k\omega t) \tag{3.8}$$

– Inserting (3.8) into (3.5) we get:

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t) e^{ik\omega t} dt \quad , \quad k=1,2,3,... \tag{3.9}$$

which is the same as (3.1) which we wanted to prove.

3.2. Calculating C_k directly from complex definition

- In this chapter we shall prove (1.9):

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} e^{-\sigma t} f(t) e^{ik\omega t} dt \quad , \quad k=1,2,3,... \tag{3.10}$$

- One way to calculate C_k is directly from (1.7) like this:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{\sigma t} e^{ik\omega t} \Big/ e^{-\sigma t} e^{-iK\omega t} \tag{3.11}$$

$$f(t)e^{-\sigma t} e^{-iK\omega t} = \sum_{k=-\infty}^{\infty} C_k e^{(\sigma t-\sigma t)} e^{i(k-K)\omega t} \tag{3.12}$$

$$= \sum_{k=-\infty}^{\infty} C_k e^{i(k-K)\omega t} \tag{3.13}$$

– Using Euler's formula (4.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{3.14}$$

to rewritte part inside sum, and integrating the whole formula, results in:

$$f(t)e^{-\sigma t} e^{-iK\omega t} = \sum_{k=-\infty}^{\infty} C_k \{\cos[(k-K)\omega t] + j\sin[(k-K)\omega t]\} \Big/ \int_{-T/2}^{T/2} dt \tag{3.15}$$

$$\int_{-\pi}^{\pi} f(t)e^{-\sigma t} e^{-iK\omega t} dt = \sum_{k=-\infty}^{\infty} C_k \left\{ \int_{-T/2}^{T/2} \cos[(k-K)\omega t] dt + \int_{-T/2}^{T/2} \sin[(k-K)\omega t] dt \right\} \tag{3.16}$$

– Because of (4.11) and (4.12):

$$\int_{-T/2}^{T/2} \cos(k\omega t) dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{3.17}$$

$$\int_{-T/2}^{T/2} \sin(k\omega t) dt = 0 \quad , \quad k \neq 0 \tag{3.18}$$

all integrals where (K+k) or (K-k) are not zero, are equal to zero, leaving integrals where k=K:

$$\int_{-T/2}^{T/2} f(t)e^{-\sigma t} e^{-iK\omega t} dt = C_K \int_{-T/2}^{T/2} \cos[(K-K)\omega t] dt + C_K \int_{-T/2}^{T/2} \sin[(K-K)\omega t] dt \tag{3.19}$$

$$= C_K \int_{-T/2}^{T/2} \cos(0\omega t) dt + C_K \int_{-T/2}^{T/2} \sin(0\omega t) dt \tag{3.20}$$

$$= C_K \int_{-T/2}^{T/2} dt \tag{3.21}$$

$$= C_K 1 \Big/_{-T/2}^{T/2} \tag{3.22}$$

$$= C_K \left(\frac{T}{2} - -\frac{T}{2} \right) \tag{3.23}$$

$$= C_K T \tag{3.24}$$

$$C_K = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-\sigma t} e^{-iK\omega t} dt \tag{3.25}$$

– Formula (3.25) shows how to calculate concrete coefficient C_K so we can rewrite it by replacing K with k:
– By finding formula for C_k we have at the same time proven that it is possible to make approximation using (3.11).

4. Equations

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \quad (4.1)$$

$$\sin(-\varphi) = -\sin(\varphi) \quad (4.2)$$

$$\cos(-\varphi) = \cos(\varphi) \quad (4.3)$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad (4.4)$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \quad (4.5)$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \quad (4.6)$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \quad (4.7)$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\cos(k_2\omega t)dt = 0 \quad \text{for all integers } k_1, k_2 \quad (4.8)$$

$$\int_{-T/2}^{T/2} \sin(k_1\omega t)\sin(k_2\omega t)dt = \begin{cases} \frac{T}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \quad (4.9)$$

$$\int_{-T/2}^{T/2} \cos(k_1\omega t)\cos(k_2\omega t)dt = \begin{cases} \frac{1}{T} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \quad (4.10)$$

$$\int_{-T/2}^{T/2} \cos(k\omega t)dt = \begin{cases} T & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \quad (4.11)$$

$$\int_{-T/2}^{T/2} \sin(k\omega t)dt = 0 \quad , \quad k \neq 0 \quad (4.12)$$

5. MATLAB

5.1. Figure 1.1.

```
%CLear all variables.
clear;

%Define graph appearance.
xl = -1;
xr = 3*pi;
yu = 8;
yd = -8;
figure(1); clf; grid on; hold on; axis([xl xr yd yu]);
line([xl xr],[0 0],'Color','k','LineWidth',3);
line([0 0],[yd yu],'Color','k','LineWidth',3);
xlabel('t','FontSize',14); ylabel('f(t)','FontSize',14);

%Define input variable.
t=[xl:0.01:xr];

%Rising signal.
T     = 1;
w1    = 2*pi/T;
fi1   = pi/2;
ro1   = 0.2;
ampl1 = exp(ro1*t);
sint1 = ampl1.*sin(w1*t+fi1);
plot(t,sint1 ,'LineWidth',2);
plot(t,ampl1 ,'Color','r');
plot(t,-ampl1,'Color','r');
title('f(t)=e^0^.^2^tsin(2\pit+\pi/2)','FontSize',14);

%Rising signal.
% T     = 1;
% w1    = 2*pi/T;
% fi1   = 0;
% ro1   = -0.2;
% ampl1 = 4*exp(ro1*t);
% sint1 = ampl1.*sin(w1*t+fi1);
% plot(t,sint1 ,'LineWidth',2);
% plot(t,ampl1 ,'Color','r');
% plot(t,-ampl1,'Color','r');
% title('f(t)=4*e^-^0^.^2^tsin(2\pit)','FontSize',14);

%Rising signal.
% T     = 1;
% w1    = 2*pi/T;
% fi1   = 0;
% ro1   = -0.2;
% ampl1 = 4;
% sint1 = ampl1.*sin(w1*t+fi1);
% plot(t,sint1 ,'LineWidth',2);
% plot(t,ampl1 ,'Color','r');
% plot(t,-ampl1,'Color','r');
% title('f(t)=4*e^0^tsin(2\pit)','FontSize',14);
% xlabel('t','FontSize',14); ylabel('f(t)','FontSize',14);
```

5.2. Figure 1.2.

```
%CLear all variables.
clear;

%Define graph appearance.
xl = -1;
xr = 3*pi;
yu = 8;
yd = -8;
figure(1); clf; grid on; hold on; axis([xl xr yd yu]);
line([xl xr],[0 0],'Color','k','LineWidth',3);
line([0 0],[yd yu],'Color','k','LineWidth',3);
xlabel('t','FontSize',14); ylabel('f(t)','FontSize',14);

%Define input variable.
t=[xl:0.01:xr];

%Components.
T     = 3;
w     = 2*pi/T;
A0    = 2;
ro    = 0.2;
fi1   = pi/2;                    fi2   = 0;                       fi3   = pi;
ampl1 = exp(ro*t);               ampl2 = 0.6*exp(ro*t);           ampl3 = 0.2*exp(ro*t);
sint1 = ampl1.*sin(w*t+fi1);     sint2 = ampl2.*sin(2*w*t+fi2);   sint3 = ampl3.*sin(3*w*t+fi3);
line([xl xr],[A0 A0],'Color','g','LineWidth',2);
plot(t,sint1 ,'Color','b','LineWidth',2);
plot(t,sint2 ,'Color','r','LineWidth',2);
plot(t,sint3 ,'Color','m','LineWidth',2);
plot(t,ampl1 ,'Color','b');      plot(t,ampl2 ,'Color','r');      plot(t,ampl3 ,'Color','m');
plot(t,-ampl1,'Color','b');      plot(t,-ampl2,'Color','r');      plot(t,-ampl3,'Color','m');
title('A_0=2, f_1(t)=e^0^.^2^tsin(\omegat+\pi/2), f_2(t)=0.6e^0^.^2^tsin(2\omegat),
f_3(t)=0.2e^0^.^2^tsin(3\omegat+\pi)','FontSize',12);

%Define graph appearance.
xl = -1;
xr = 3*pi;
yu = 8;
yd = -8;
figure(2); clf; grid on; hold on; axis([xl xr yd yu]);
line([xl xr],[0 0],'Color','k','LineWidth',3);
line([0 0],[yd yu],'Color','k','LineWidth',3);

%Combination.
title('f(t)=A_0+2e^0^.^2^tsin(\omegat+\pi/2)+e^0^.^2^tsin(2\omegat)+0.2e^0^.^2^tsin(3\omegat+\
pi)','FontSize',14);
plot(t,A0+sint1+sint2+sint3 ,'Color','b','LineWidth',2);
xlabel('t','FontSize',14); ylabel('f(t)','FontSize',14);
```

6. References

[1] Trigonometric Functions.doc

[2] Mean Value.doc

