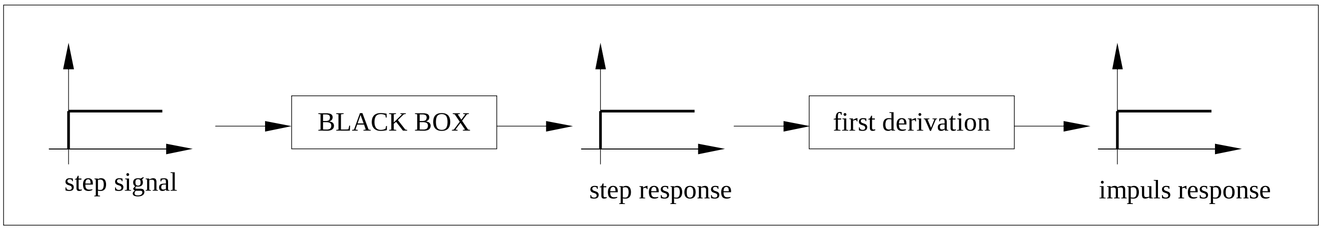
Convolution

1. Convolution

- Convolution is single most important thing in signal processing.
- Convolution allows us to calculate output signal from system if we know:
  - input signal into system
  - output signal from the system when input signal into system is impuls.
- The main idea behind convolution is to aproximate input signal as sum of multiple step or impuls signals.
- Then we calculate output signal for each of these input step or impuls signals and add the results.

g(t-τ) – value of output signal at time t if input signal was step signal which started at t=τ.
h(t-τ) – value of output signal at time t if input signal was impuls signal at t=τ.

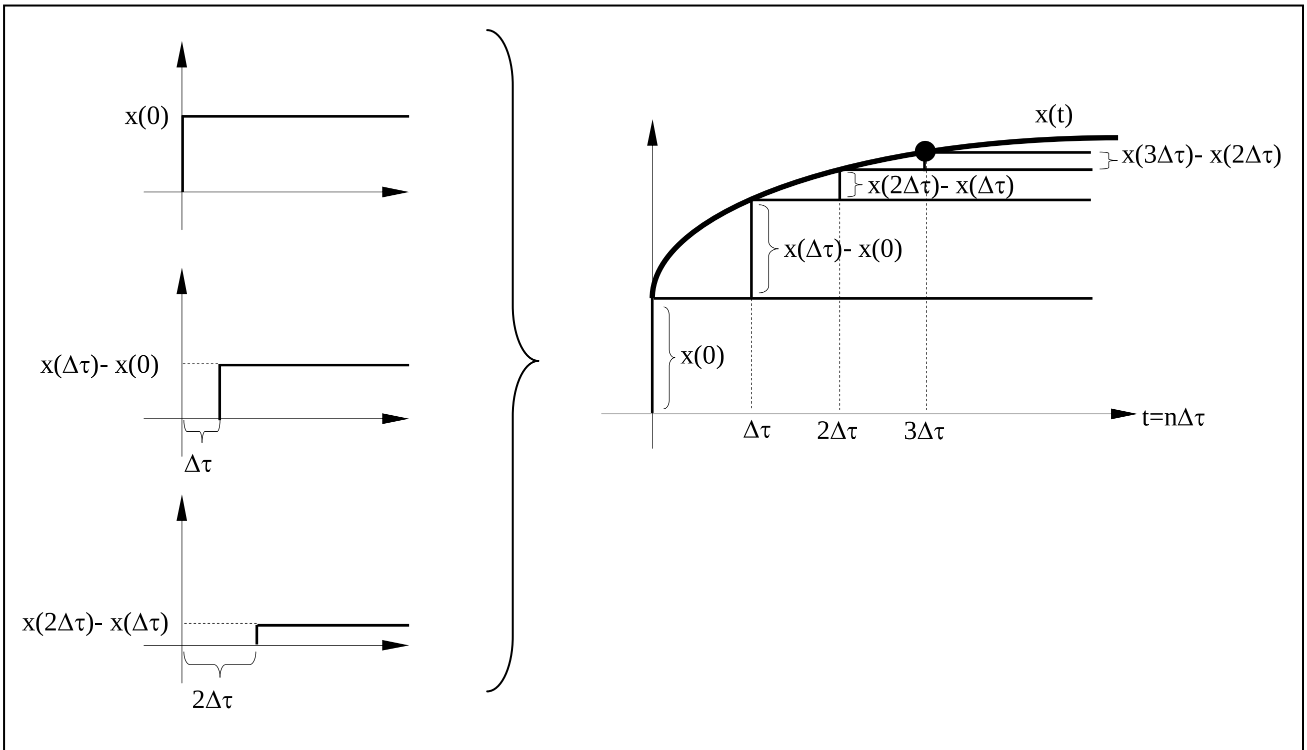
- You go into the store and buy some black box containing some system.
- Then you put step signal into the input of the system and observe the output called step response.
- Then you put this output into some system that does firts derivation.
- Resulting signal is called impulse response.
- This is how the output of the system would look like if we put impuls function at the input
- But since it is not possible to practicaly create impuls function in order to place her at the input we had to use the above procedure to get the impuls response.
- All of this is ilustrated on following figure.



- Once you have

1.1. Aproximating input signal as sum of step signals

- In this approach we aproximate input signal $x_u(t)$ as sum od multiple step signals like shown on following figure:



$$y(t) \approx x(0)g(t) + [x(\Delta t) - x(0)]g(t - \Delta t) + [x(2\Delta t) - x(\Delta t)]g(t - 2\Delta t) + [x(3\Delta t) - x(2\Delta t)]g(0)$$

Figure 1.1. Aproximating input signal as sum of step signals.

– We can now aproximate y(t), output signal at time t, like this:

$$y(t) \approx x(0)g(t-0) + [x(\Delta t) - x(0)]g(t-\Delta t) + [x(2\Delta t) - x(\Delta t)]g(t-2\Delta t) + ... + [x(n\Delta t) - x((n-1)\Delta t)]g(t-n\Delta t) \quad (1.1)$$

– If we use following substitution:

$$x'(k\Delta t) = \frac{x(k\Delta t) - x((k-1)\Delta t)}{\Delta t} \quad (1.2)$$

we can rewritte (1.1) like:

$$y(t) \approx x(0)g(t-0) + x'(\Delta t)\Delta t g(t-\Delta t) + x'(2\Delta t)\Delta t g(t-2\Delta t) + ... + x'(n\Delta t)\Delta t g(t-n\Delta t) \quad (1.3)$$

$$y(t) \approx x(0)g(t) + \sum_{k=1}^{n} x'(k\Delta\tau)\cdot\Delta\tau\cdot g(t-k\Delta\tau) \quad (1.4)$$

– If we let $\Delta\tau$ become close to zero we get:

$$y(t) = x(0)g(t) + \lim_{\Delta\tau\to 0}\sum_{k=1}^{t/\Delta\tau} x'(k\Delta\tau)\cdot\Delta\tau\cdot g(t-k\Delta\tau)$$

$$y(t) = x(0)g(t) + \sum_{k=1}^{t/d\tau} x'(kd\tau)\cdot d\tau\cdot g(t-kd\tau)$$

$$y(t) = x(0)g(t) + \int_0^t x'(\tau)g(t-\tau)d\tau$$

$$x'(\tau)d\tau = dv \;\Big/ \int \qquad g(t-\tau) = u \;\Big/ \frac{d}{d\tau}$$

$$x(\tau) = v \qquad -g'(t-\tau) = \frac{du}{d\tau}$$

$$y(t) = x(0)g(t) + g(t-\tau)x(\tau)\Big/_0^t + \int_0^t x(\tau)g'(t-\tau)d\tau$$

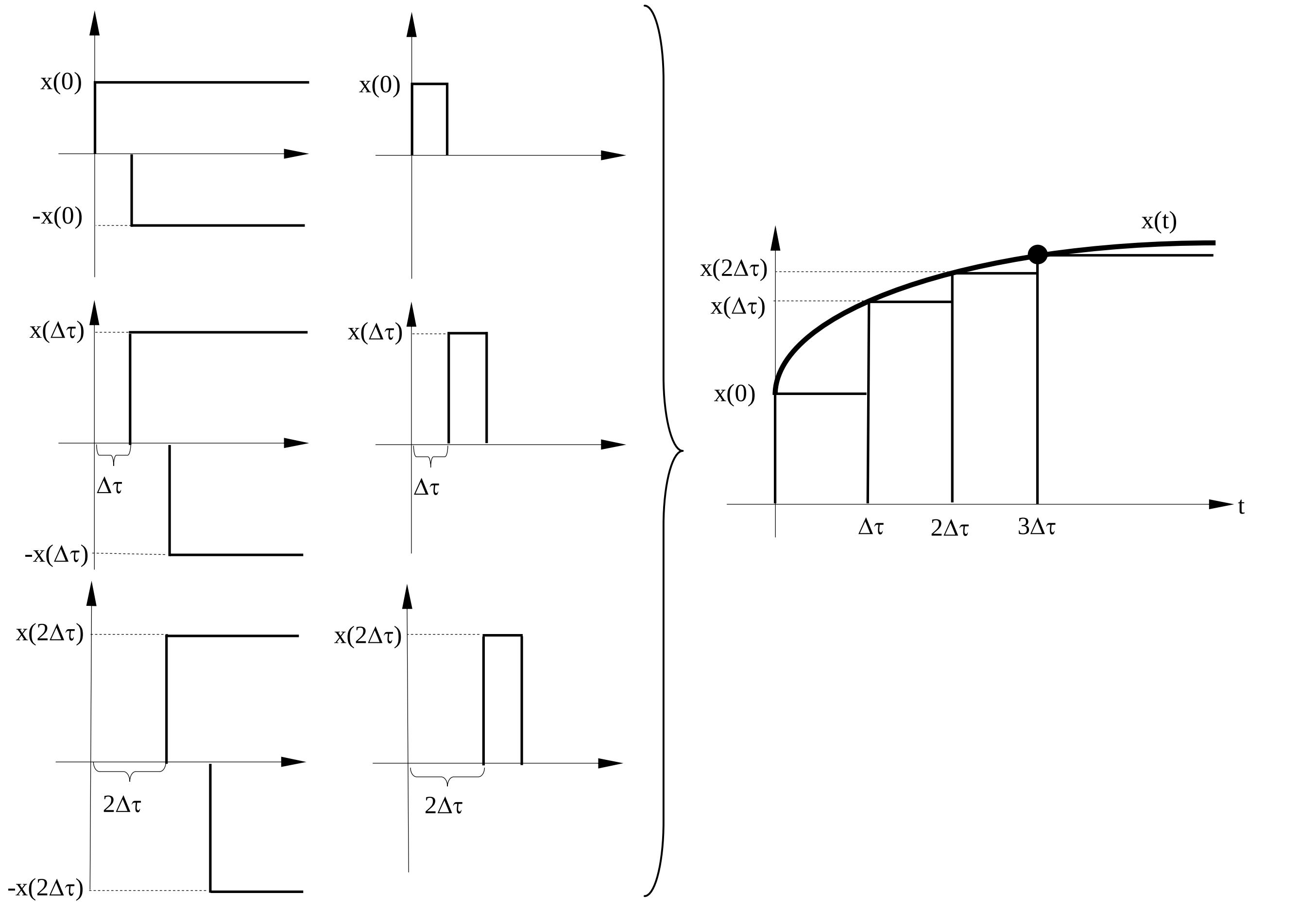
$$y(t) = x(0)g(t) + g(0)x(t) - g(t)x(0) + \int_0^t x(\tau)g'(t-\tau)d\tau$$

$$y(t) = x(t)g(0) + \int_0^t x(\tau)g'(t-\tau)d\tau$$

$$\boxed{y(t) = x(t)g(0) + \int_0^t x(\tau)h(t-\tau)d\tau}$$

1.2. Aproximating input signal as sum of aproximated impuls signals

– In this approach we aproximate input signal x(t) as sum od multiple step signals like shown on following figure:



$$y(t) \approx \left[x(0)g(t-0) - x(0)g(t-\Delta\tau)\right] + \left[x(\Delta\tau)g(t-\Delta\tau) - x(\Delta\tau)g(t-2\Delta\tau)\right] + \left[x(2\Delta\tau)g(t-2\Delta\tau) - x(2\Delta\tau)g(t-3\Delta\tau\right] + x(3\Delta\tau)g(0)$$

Figure 1.2 Aproximating input signal as sum of aproximated impuls signals.

– We can now aproximate y(t), output signal at time t, like this:

$$y(t) \approx [x(0)g(t-0) - x(0)g(t-\Delta\tau)] + [x(\Delta\tau)g(t-\Delta\tau) - x(\Delta\tau)g(t-2\Delta\tau)] + [x(2\Delta\tau)g(t-2\Delta\tau) - x(2\Delta\tau)g(t-3\Delta\tau] + \cdots + x(n\Delta\tau)g(t-n\Delta\tau) \tag{1.5}$$

$$y(t) \approx x(0)[g(t-0) - g(t-\Delta\tau)] + x(\Delta\tau)[g(t-\Delta\tau) - g(t-2\Delta\tau)] + x(2\Delta\tau)[g(t-2\Delta\tau) - g(t-3\Delta\tau] + \cdots + x(n\Delta\tau)g(t-n\Delta\tau) \tag{1.6}$$

– Formula (1.6) is exactly the same as (1.1) with rearanged elements.

– Since first derivative of step response is impuls response we can write:

$$h(t) \approx \frac{g(t) - g(t-\Delta\tau)}{\Delta\tau} \tag{1.7}$$

– Inserting (1.7) into (1.6) we get:

$$y(t) \approx x(0)\Delta\tau h(t) + x(\Delta\tau)\Delta\tau h(t-\Delta\tau) + x(2\Delta\tau)\Delta\tau h(t-2\Delta\tau) + \ldots + x(t)g(0) \tag{1.8}$$

$$y(t) \approx x(t)g(0) + \sum_{k=0}^{n} x(k\Delta\tau)\cdot\Delta\tau\cdot h(t-k\Delta\tau) \tag{1.9}$$

$$y(t) = x(t)g(0) + \lim_{\Delta\tau\to 0} \sum_{k=0}^{n=t/\Delta\tau} x(k\Delta\tau)\cdot\Delta\tau\cdot h(t-k\Delta\tau) \tag{1.10}$$

$$y(t) = x(t)g(0) + \sum_{k=0}^{t/d\tau} x(kd\tau)\cdot d\tau\cdot h(t-kd\tau) \tag{1.11}$$

$$\boxed{y(t) = x(t)g(0) + \int_0^t x(\tau)h(t-\tau)d\tau} \tag{1.12}$$