DFS - Descrete Fourier Sequence - Part I

1. Overview

- Problem definition.

– Discrete Fourier Sequence is a way of representing descrete periodical signal with infinite sum of descrete sinus signals. Descrete signal is defined with folowing values:

t	–	discrete time index t=0,1,2,... (this is not time)
f(t)	–	signal value at descrete time step t=0,1,2,...
N	–	number of descrete time steps in one interval
T/N	–	sampling period, time that passes between 2 samples
ω	–	frequency of the signal, how many radians pass during 1 sampling period

- Different forms of DFS.

– Equations for DFS using A_k and φ_k are as follows:

$$f(n) = A_0 + \sum_{k=1}^{N/2} A_k \sin(k\psi n + \varphi_k) \tag{1.1}$$

– Equations for DFS using a_k and b_k are as follows:

$$f(n) = a_0 + \sum_{k=1}^{N/2} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] \tag{1.2}$$

$$a_0 = \frac{1}{N}\sum_{n=1}^{N} f(n) \qquad \text{(signal's mean value)} \tag{1.3}$$

$$a_k = \frac{2}{N}\sum_{n=1}^{N} f(n)\cos(k\psi n) \quad , \quad k=1,2,\dots \text{ till } N/2-1 \text{ for even N, or } (N-1)/2 \text{ for odd N} \tag{1.4}$$

$$b_k = \frac{2}{N}\sum_{n=1}^{N} f(n)\sin(k\psi n) \quad , \quad k=1,2,\dots \text{ till } N/2-1 \text{ for even N, or } (N-1)/2 \text{ for odd N} \tag{1.5}$$

– Equations for DFS using C_k are as follows:

$$f(n) = C_0 + \sum_{k=1}^{N} C_k e^{ik\psi n} \tag{1.6}$$

$$C_0 = \frac{1}{N}\sum_{k=1}^{N} f(n) \tag{1.7}$$

$$C_k = \frac{1}{N}\sum_{k=1}^{N} f(n) e^{-ik\psi n} \quad , \quad k=1,\dots,N-1 \tag{1.8}$$

- Relations between coefficients of different forms.

– Calculating coefficients of the second form from coefficients of the main form:

$$a_0 = A_0 \tag{1.9}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.10}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.11}$$

– Calculating coefficients of the main form from coefficients of the second form:

$$A_0 = a_0 \tag{1.12}$$

$$A_k = \sqrt{a_k^2 + b_k^2} \quad , \quad k=1,2,... \tag{1.13}$$

$$\varphi_k = \text{cotan}\left(\frac{a_k}{b_k}\right) \tag{1.14}$$

– Calculating coefficients of the complex form from coefficients of the second form:

$$C_0 = a_0 \tag{1.15}$$

$$C_k = \frac{1}{2}\left(a_k - ib_k\right) \quad , \quad k=1,2,... \tag{1.16}$$

1.1. Transforming into second form

- In this chapter we will prove (1.2):

$$f(n) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n) \right] \tag{1.17}$$

- Extracting A_k from (1.1) is easy, but extracting φ_k isn't:

$$f(n) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\psi n + \varphi_k) \tag{1.18}$$

– That is why we shall transform (1.18) into form in which unknowns are better separated for easier extraction.

– Using (4.6):

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

terms under sum in (1.18) can be transformed like this:

$$\begin{aligned} A_k \sin(k\psi n + \varphi_k) &= A_k \sin(k\psi n)\cos(\varphi_k) + A_k \cos(k\psi n)\sin(\varphi_k) \\ &= A_k \cos(\varphi_k)\sin(k\psi n) + A_k \sin(\varphi_k)\cos(k\psi n) \\ &= A_k \sin(\varphi_k)\cos(k\psi n) + A_k \cos(\varphi_k)\sin(k\psi n) \end{aligned} \tag{1.19}$$

– We continue by introducing following substitutions:

$$a_0 = A_0 \tag{1.20}$$

$$a_k = A_k \sin(\varphi_k) \tag{1.21}$$

$$b_k = A_k \cos(\varphi_k) \tag{1.22}$$

– Inserting (1.21) and (1.22) into (1.19), we get:

$$A_k \sin(k\psi n + \varphi_k) = a_k \cos(k\omega t) + b_k \sin(k\omega t) \tag{1.23}$$

– Inserting (1.20) and (1.23) into (1.18) we get (1.17):

$$f(n) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n) \right] \tag{1.24}$$

which we wanted to prove.

- We will end this chapter by showing how to calculate A_0, A_k and φ_k from a_0, a_k and b_k.

– From (1.20) we can simply write:

$$A_0 = a_0 \tag{1.25}$$

– Amplitude A_k can be calculated by combining (1.21) and (1.22) like this:

$$a_k^2 + b_k^2 = A_k^2[\sin^2(\varphi) + \cos^2(\varphi)]$$

$$a_k^2 + b_k^2 = A_k^2$$

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{1.26}$$

– Phase shift φ_k can be calculated again by combining (1.21) and (1.22) like this:

$$\frac{a_k}{b_k} = \frac{A_k \sin(\varphi_k)}{A_k \cos(\varphi_k)}$$

$$= \tan(\varphi_k)$$

$$\varphi_k = \text{cotan}\left(\frac{a_k}{b_k}\right) \tag{1.27}$$

1.2. Transforming into complex form

- In this chapter we shall prove equation (1.6):

$$f(n) = C_0 + \sum_{k=1}^{N} C_k e^{ik\psi t} \tag{1.28}$$

- We start with equation (1.2):

$$f(n) = a_0 + \sum_{k=1}^{N/2} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] \tag{1.29}$$

– We continue by using following relation which will be proven shortly:

$$\sum_{k=1}^{N/2} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] = \frac{1}{2}\sum_{k=1}^{N} (a_k - ib_k)\left[\cos(k\psi n) + i\sin(k\psi n)\right] \tag{1.30}$$

– Inserting (1.30) into (1.29) we get:

$$f(n) = a_0 + \frac{1}{2}\sum_{k=1}^{N} (a_k - ib_k)\left[\cos(k\psi n) + i\sin(k\psi n)\right] \tag{1.31}$$

– Using following substitutions:

$$C_0 = a_0 \tag{1.32}$$

$$C_k = \frac{1}{2}(a_k - ib_k) \tag{1.33}$$

and Euler's formula (4.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{1.34}$$

equation (1.31) can be rewritten as:

$$f(n) = C_0 + \sum_{k=1}^{N} C_k e^{ik\psi t} \tag{1.35}$$

which we wanted to prove.

- And now to prove (1.30):

$$\sum_{k=1}^{N/2}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] = \frac{1}{2}\sum_{k=1}^{N}\left(a_k - ib_k\right)\left[\cos(k\psi n) + i\sin(k\psi n)\right] \quad (1.36)$$

– By calculating the right side of (1.36) we get:

$$\sum_{k=1}^{N/2}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] = \frac{1}{2}\sum_{k=1}^{N}\left\{\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] + i\left[a_k \sin(k\psi n) - b_k \cos(k\psi n)\right]\right\}$$

$$= \frac{1}{2}\sum_{k=1}^{N}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] + i\frac{1}{2}\sum_{k=1}^{N}\left[a_k \sin(k\psi n) - b_k \cos(k\psi n)\right] \quad (1.37)$$

– To calculate two sums in (1.37) following remarks have to be made.

– If we take one exact value for k, for instance k=-K, and put it into (1.4) and (1.5), using (4.4) and (4.5):

$$\sin((N\text{-}k)\psi n) = \sin(\text{-}k\psi n) \quad (1.38)$$

$$\cos((N\text{-}k)\psi n) = \cos(\text{-}k\psi n) \quad (1.39)$$

we get:

$$a_{N-K} = \frac{2}{N}\sum_{n=1}^{N} f(n)\cos\left[(N-k)\psi n\right]$$

$$= \frac{2}{N}\sum_{n=1}^{N} f(n)\cos(k\psi n)$$

$$= a_K \quad (1.40)$$

$$b_{N-K} = \frac{2}{N}\sum_{n=1}^{N} f(n)\sin\left[(N-k)\psi n\right]$$

$$= -\frac{2}{N}\sum_{n=1}^{N} f(n)\sin(k\psi n)$$

$$= -b_K \quad (1.41)$$

– We will now repeat last four relations (1.38), (1.39), (1.40), (1.41) to have them all neatly at one place:

$$\sin\left[(N-K)\psi n\right] = -\sin(K\psi n)$$

$$\cos\left[(N-K)\psi n\right] = \cos(K\psi n)$$

$$a_{N-K} = a_K \quad (1.42)$$

$$b_{N-K} = -b_K \quad (1.43)$$

– Using this four relations we will now calculate the result of summing two summands from first sum in (1.37), where first summand is for k=K and second for k=-K:

$$\left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] + \left\{a_{N-K} \cos\left[(N-K)\omega t\right] + b_{N-K} \sin\left[(N-K)\omega t\right]\right\} =$$

$$= \left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] + \left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] =$$

$$= 2\left[a_K \cos(K\omega t) + b_K \sin(K\omega t)\right] \quad (1.44)$$

– Using same method for the second sum in (1.37), we get:

$$\left[a_K \sin(K\omega t) - b_K \cos(K\omega t)\right] + \left\{a_{N-K} \sin\left[(N-K)\omega t\right] - b_{N-K} \cos\left[(N-K)\omega t\right]\right\} =$$

$$= \left[a_K \sin(K\omega t) - b_K \cos(K\omega t)\right] + \left[-a_K \sin(K\omega t) + b_K \cos(K\omega t)\right] =$$

$$= 0 \qquad (1.45)$$

- From (1.42) we conclude that terms of first sum in (1.37) are symmetric around k=N/2 since terms with K and -K have the same values allowing us to rewrite fhat first sum like this:

$$\sum_{k=1}^{N}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] = 2\sum_{k=1}^{N/2}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] \qquad (1.46)$$

- From (1.43) we conclude that terms of second sum in (1.37) are also symetryc around k=0 but with different sign, since terms with K and -K have the same absolute values but different sign.
- This means that these terms are being canceled inside the sum making the whole sum equal to zero:

$$\sum_{k=1}^{N}\left[a_k \sin(k\psi n) - b_k \cos(k\psi n)\right] = 0 \qquad (1.47)$$

- Inserting (1.46) and (1.47) into (1.37) we get right side becomes equal to the left one ending our proof of (1.36).

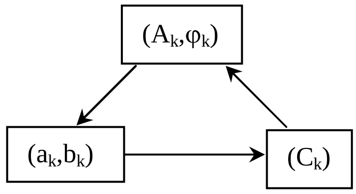
$$\sum_{k=1}^{N/2}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] = \frac{1}{2}2\sum_{k=1}^{N/2}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right] + i\frac{1}{2}0$$

$$= \sum_{k=1}^{N/2}\left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]$$

1.3. Additional observations

- Connecting coefficients

– The whole procedure could be summarized like this.
– To calculate A_0, A_k and φ_k first we calculate a_0, a_k and b_k.
– From them we construct C_k which we use to calculate A_k and φ_k.
– This can be presented with following triangle:



– Parameters A_0, A_k and φ_k are calculated by scaning the original signal f(n).
– Each scan is done by multiplying the signal with sine or cosine of $k\omega$ frequency and integrating the result over one period which gives us information of the amplitude and phase of the sine signal we were scanning for.

- How many coefficients to calculate

– For odd N, both coefficients a_k and b_k are different till k=(N-1)/2.
– For even N, coefficients a_k are different till k=N/2 and b_k till N/2-1.
But for $a_{N/2}$ formula (1.4) gives double the actual amplitude of cosine component.
This is why for even N both coefficients are calculated only till N/2-1.

2. Calculating a_k and b_k

- In this chapter we will show how to calculate a_0, a_k and b_k from (1.2).

$$f(n) = a_0 + \sum_{k=1}^{N/2} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n) \right] \qquad (2.1)$$

- How could we calculate some concrete coeficient a_K, where k=K?

– Idea is to multiply the equation (2.1) with something which would kill all a_k and b_k, which are unknowns in (2.1), but it wouldn't kill a_K, which we want to calculate.
– This way a_K will be left with things which are known and we would be able to calculate it.
– Each a_k and b_k should have its own thing with which we should multiply (2.1).

- To achieve this we shall use following properties of sine and cosine signals (4.17), (4.18) and (4.19):

$$\sum_{n=1}^{N} \sin(k_1\psi n)\cos(k_2\psi n) = 0 \quad \text{for all integers } k_1, k_2 \qquad (2.2)$$

$$\sum_{n=1}^{N} \sin(k_1\psi n)\sin(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \qquad (2.3)$$

$$\sum_{n=1}^{N} \cos(k_1\psi n)\cos(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \qquad (2.4)$$

– If you multiply sine signal with another sine or cosine signal of frequency k times greater or smaller, or with the cosine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (2.3) only if you multiply sine signal with another sine signal of the same frequency you will get result different then zero.

– Same way if you multiply cosine signal with another sine or cosine signal of frequency k times greater or smaller, or with the sine signal of the same frequency, and sum the result over one period, you will get zero.
– According to (2.4) only if you multiply cosine signal with another cosine signal of the same frequency you will get result different then zero.

- Calculating a_0:

$$f(n) = a_0 + \sum_{k=1}^{N/2} \left[a_k \cos(k\psi nt) + b_k \sin(k\psi n) \right] \Big/ \sum_{n=1}^{N} \qquad (2.5)$$

$$\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} a_0 = a_0 \sum_{n=1}^{N} 1 = a_0 N \qquad (2.6)$$

- Calculating a_k

$$f(n) = a_0 + \sum_{k=1}^{N/2} \left[a_k \cos(k\psi n) + b_k \sin(k\psi n) \right] \Big/ \cos(K\psi n) \Big/ \sum_{n=1}^{N} \qquad (2.7)$$

$$\sum_{n=1}^{N} f(n)\cos(K\psi n) = a_K \sum_{t=0}^{N} \cos(K\psi n)\cos(K\psi n) = a_K \frac{N}{2} \quad , \ k=1,2,... \qquad (2.8)$$

- Calculating b_k:

$$f(n) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t) \right] \Big/ \sin(K\omega t) \Big/ \sum_{t=0}^{N-1} \qquad (2.9)$$

$$\sum_{n=1}^{N} f(n)\sin(K\psi n) = b_K \sum_{n=1}^{N} \sin(K\psi n)\sin(K\psi n) = b_K \frac{N}{2} \quad , \ k=1,2,... \qquad (2.10)$$

2.1. Calculating a_0

- In this chapter we will prove (1.3):

$$a_0 = \frac{1}{N}\sum_{n=1}^{N} f(n) \tag{2.11}$$

- To calculate a_0 we must multiply equation (2.11) like this:

$$f(n) = a_0 + \sum_{k=1}^{N/2}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] \Big/ \sum_{n=1}^{N} \tag{2.12}$$

$$\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} a_0 + \sum_{k=1}^{N/2}\left[\sum_{n=1}^{N} a_k \cos(k\psi n) + \sum_{n=1}^{N} b_k \sin(k\psi n)\right] \tag{2.13}$$

– Becuase of (4.13):

$$\sum_{n=1}^{N} \cos(k\psi n) = \begin{cases} N & \text{for } k = 0 \\ 0 & \text{for integer } k \neq 0 \end{cases} \tag{2.14}$$

all terms defined with second integral on the right side are zero.

– Becuase of (4.14):

$$\sum_{n=1}^{N} \sin(k\psi n) = 0 \quad \text{for all integer } k \tag{2.15}$$

all terms defined with second integral on the right side are zero.

– This leaves us with:

$$\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} a_0 \tag{2.16}$$

$$= Na_0 \tag{2.17}$$

$$a_0 = \frac{1}{N}\sum_{n=1}^{N} f(n) \tag{2.18}$$

which is the same as (2.11) which we wanted to prove.

2.2. Calculating a_k

- In this chapter we will prove (1.4):

$$a_k = \frac{2}{N}\int_{-T/2}^{T/2} f(t)\cos(k\psi n)dt \tag{2.19}$$

- To calculate some concrete coefficient a_K we must multiply equation (2.19) like this:

$$f(n) = a_0 + \sum_{k=1}^{N/2}\left[a_k\cos(k\psi n) + b_k\sin(k\psi n)\right] \Big/ \cos(K\psi n) \tag{2.20}$$

$$f(n)\cos(K\psi n) = a_0\cos(K\psi n) + \sum_{k=1}^{N/2}\left[a_k\cos(k\psi n)\cos(K\psi n) + b_k\sin(k\psi n)\cos(K\psi n)\right] \Big/ \sum_{n=1}^{N} \tag{2.21}$$

$$\sum_{n=1}^{N} f(n)\cos(K\psi n) = \sum_{n=1}^{N} a_0\cos(K\psi n) + \sum_{k=1}^{N/2}\left[\sum_{n=1}^{N} a_k\cos(k\psi n)\cos(K\psi n) + \sum_{n=1}^{N} b_k\sin(k\psi n)\cos(K\psi n)\right] \tag{2.22}$$

- Detecting terms equal to zero.

– Since K>0, becuase of (4.13):

$$\sum_{n=1}^{N}\cos(k\psi n) = \begin{cases} N & \text{for } k = 0 \\ 0 & \text{for integer } k \neq 0 \end{cases} \tag{2.23}$$

first integral on the right side is zero.

– Becuse of (4.17):

$$\sum_{n=1}^{N}\sin(k_1\psi n)\cos(k_2\psi n) = 0 \quad \text{for all integers } k_1, k_2 \tag{2.24}$$

all terms defined with last integral on the right side are also zero.

– Because of (4.19):

$$\sum_{n=1}^{N}\cos(k_1\psi n)\cos(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.25}$$

all terms defined with second integral on the right side are zero for k≠K leaving only one term when k=K:

$$\sum_{n=1}^{N} f(n)\cos(K\psi n) = \sum_{n=1}^{N} a_K\cos(K\omega t)\cos(K\omega t) \tag{2.26}$$

- Again because of (4.19) integral on the right side is N/2:

$$\sum_{n=1}^{N} f(n)\cos(K\psi n) = a_K\frac{N}{2} \tag{2.27}$$

$$a_K = \frac{2}{N}\sum_{n=1}^{N} f(n)\cos(K\psi n) \tag{2.28}$$

which is the same as (2.19) which we wanted to prove.

2.3. Calculating b_k

- In this chapter we will prove (1.5):

$$b_k = \frac{2}{N}\sum_{n=1}^{N} f(t)\sin(k\psi n) \quad , \quad k=1,2,... \tag{2.29}$$

– To calculate some concrete coefficient b_K we must multiply equation (2.29) like this:

$$f(n) = a_0 + \sum_{k=1}^{N/1}\left[a_k \cos(k\psi n) + b_k \sin(k\psi n)\right] \Big/ \sin(K\psi n) \tag{2.30}$$

$$f(n)\sin(K\psi n) = a_0 + \sum_{k=1}^{N}\left[a_k \sin(K\psi n)\cos(k\psi n) + b_k \sin(K\psi n)\sin(k\psi n)\right] \Big/ \sum_{n=1}^{N} \tag{2.31}$$

$$\sum_{n=1}^{N} f(n)\sin(K\psi n) = \sum_{n=1}^{N} a_0 + \sum_{k=1}^{N}\left[\sum_{n=1}^{N} a_k \sin(K\psi n)\cos(k\psi n) + \sum_{n=1}^{N} b_k \sin(K\psi n)\sin(k\psi n)\right] \tag{2.32}$$

- Detecting terms equal to zero.

– Since K>0, becuase of (4.14):

$$\sum_{n=1}^{N} \sin(k\psi n) = 0 \quad \text{for all integer } k \tag{2.33}$$

first integral on the right side is zero.

– Becuse of (4.17):

$$\sum_{n=1}^{N} \sin(k_1\psi n)\cos(k_2\psi n) = 0 \quad \text{for all integers } k_1, k_2 \tag{2.34}$$

all terms defined with secobd integral on the right side are also zero.

– Because of (4.18):

$$\sum_{n=1}^{N} \sin(k_1\psi n)\sin(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{2.35}$$

all terms defined with last integral on the right side are zero for k≠K which leaves only one term when k=K:

$$\sum_{n=1}^{N} f(n)\sin(K\psi n) = \sum_{n=1}^{N} b_K \sin(K\psi n)\sin(K\psi n) \tag{2.36}$$

- Again becuase of (4.18) integral on the right side is T:

$$\sum_{n=1}^{N} f(n)\sin(K\psi n) = b_K \frac{N}{2} \tag{2.37}$$

$$b_K = \frac{2}{N}\sum_{n=1}^{N} f(n)\sin(K\psi n) \tag{2.38}$$

3. Calculating C_k

– In this chapter we shall present two ways to caclulate C_k.

3.1. Calculating C_k from a_k and b_k

- In this chapter we shall prove (1.8):

$$C_k = \frac{1}{N}\sum_{k=1}^{N} f(t) e^{-ik\psi n} \tag{3.1}$$

- Coefficients C_k can be calculated from a_k and b_k using relation (1.16):

$$C_k = \frac{1}{2}\left(a_k - ib_k\right) \tag{3.2}$$

– Inserting (1.4) and (1.5):

$$a_k = \frac{2}{N}\sum_{n=1}^{N} f(t)\cos\left(k\psi n\right) \quad , \quad k=1,2,... \tag{3.3}$$

$$b_k = \frac{2}{N}\sum_{n=1}^{N} f(t)\sin\left(k\psi n\right) \quad , \quad k=1,2,... \tag{3.4}$$

into (3.2) we get:

$$C_k = \frac{1}{2}\left[\frac{2}{N}\sum_{n=1}^{N} f(t)\cos\left(k\psi n\right) - i\frac{2}{N}\sum_{n=1}^{N} f(t)\sin\left(k\psi n\right)\right]$$

$$= \frac{1}{N}\sum_{n=1}^{N} f(t)\left[\cos\left(k\psi n\right) - i\sin\left(k\psi n\right)\right] \tag{3.5}$$

– Using Euler's formula (4.1):

$$e^{i\varphi} = \cos\left(\varphi\right) + i\sin\left(\varphi\right) \tag{3.6}$$

when φ=-kωt we get:

$$e^{-ik\psi n} = \cos\left(-k\psi n\right) + i\sin\left(-k\psi n\right) \tag{3.7}$$

$$= \cos\left(k\psi n\right) - i\sin\left(k\psi n\right) \tag{3.8}$$

– Inserting (3.8) into (3.5) we get:

$$C_k = \frac{1}{N}\sum_{k=1}^{N} f(t) e^{-ik\psi n} \tag{3.9}$$

which is the same as (3.1) which we wanted to prove.

3.2. Calculating C_k directly from complex definition

- In this chapter we shall prove (1.8):

$$C_k = \frac{1}{N}\sum_{k=1}^{N} f(t)e^{-ik\psi n} \tag{3.10}$$

- One way to calculate C_k is directly from (1.6) like this:

$$f(n) = C_0 + \sum_{k=1}^{N} C_k e^{ik\psi n} \Big/ e^{-iK\psi n} \tag{3.11}$$

$$f(n)e^{-iK\psi n} = \sum_{k=1}^{N} C_k e^{i(k-K)\psi n} \tag{3.12}$$

$$= \sum_{k=1}^{N} C_k e^{i(k-K)\psi n} \tag{3.13}$$

– Using Euler's formula (4.1):

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{3.14}$$

to rewritte part inside sum, and then sum the whole formula, results in:

$$f(n)e^{-iK\psi n} = \sum_{k=1}^{N} C_k \{\cos[(k-K)\psi n] + j\sin[(k-K)\psi n]\} \Big/ \sum_{n=1}^{N} \tag{3.15}$$

$$\sum_{n=1}^{N} f(\psi n)e^{-iK\psi n}dt = \sum_{k=1}^{N} C_k \left\{\sum_{n=1}^{N}\cos[(k-K)\psi n] + \sum_{n=1}^{N}\sin[(k-K)\psi n]\right\} \tag{3.16}$$

– Because of (4.13) and (4.14):

$$\sum_{n=1}^{N}\cos(k\psi n) = \begin{cases} N & \text{for } k = 0 \\ 0 & \text{for integer } k \neq 0 \end{cases} \tag{3.17}$$

$$\sum_{n=1}^{N}\sin(k\psi n) = 0 \quad \text{for all integer k} \tag{3.18}$$

all integrals where (K+k),(K-k)≠0, are equal to zero, leaving integrals with k=K:

$$\sum_{n=1}^{N} f(\psi n)e^{-iK\psi n}dt = C_k\sum_{n=1}^{N}\cos[(K-K)\psi n] + C_k\sum_{n=1}^{N}\sin[(K-K)\psi n] \tag{3.19}$$

$$= C_k\sum_{n=1}^{N}\cos(0\psi n) + C_k\sum_{n=1}^{N}\sin(0\psi n) \tag{3.20}$$

$$= C_k\sum_{n=1}^{N}1 + C_k\sum_{n=1}^{N}0 \tag{3.21}$$

$$= C_k N + C_k 0 \tag{3.22}$$

$$= C_K N \tag{3.23}$$

$$C_K = \frac{1}{N}\sum_{n=1}^{N} f(\psi n)e^{-iK\psi n}dt \tag{3.24}$$

– Formula (3.24) shows how to calculate concrete coefficient C_K so we can rewrite it by replacing K with k:
– By finding formula for C_k we have at the same time proven that it is possible to make approximation using (3.11).

4. Equations

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \tag{4.1}$$

$$\sin(-\varphi) = -\sin(\varphi) \tag{4.2}$$

$$\cos(-\varphi) = \cos(\varphi) \tag{4.3}$$

$$\sin((N\text{-}k)\omega t) = \sin(\text{-}k\omega t) \tag{4.4}$$

$$\cos((N\text{-}k)\omega t) = \cos(\text{-}k\omega t) \tag{4.5}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{4.6}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{4.7}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \tag{4.8}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \tag{4.9}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} \tag{4.10}$$

$$\int_{-\pi}^{\pi}\cos(k\varphi)d(\varphi) = 0 \quad , \quad k\neq 0 \tag{4.11}$$

$$\int_{-\pi}^{\pi}\sin(k\varphi)d(\varphi) = 0 \quad , \quad k\neq 0 \tag{4.12}$$

$$\sum_{n=1}^{N}\cos(k\psi n) = \begin{cases} N & \text{for } k=0 \\ 0 & \text{for integer } k\neq 0 \end{cases} \tag{4.13}$$

$$\sum_{n=1}^{N}\sin(k\psi n) = 0 \quad \text{for all integer } k \tag{4.14}$$

$$\sum_{t=0}^{N-1}\cos^2(k\omega t) = \begin{cases} N & \text{for } k=0 \\ \frac{N}{2} & \text{for } k\neq 0 \end{cases} \tag{4.15}$$

$$\sum_{t=0}^{N-1}\sin^2(k\omega t) = \begin{cases} 0 & \text{for } k=0 \\ \frac{N}{2} & \text{for } k\neq 0 \end{cases} \tag{4.16}$$

$$\sum_{n=1}^{N}\sin(k_1\psi n)\cos(k_2\psi n) = 0 \quad \text{for all integers } k_1, k_2 \tag{4.17}$$

$$\sum_{n=1}^{N}\sin(k_1\psi n)\sin(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{4.18}$$

$$\sum_{n=1}^{N}\cos(k_1\psi n)\cos(k_2\psi n) = \begin{cases} \frac{N}{2} & \text{for all integers } k_1 = k_2 \text{ where } k_1, k_2 \neq 0 \\ 0 & \text{for all integers } k_1 \neq k_2 \text{ where } k_1, k_2 \geq 0 \end{cases} \tag{4.19}$$

5. References

[1] Trigonometric Functions.doc

[2] http://zone.ni.com/devzone/cda/tut/p/id/4844#toc4