Digital filters

1. Digital Filters

1.1. Introduction

- Digital filter is computer program which transforms input signal in the way that it changes it's spectrum.
- We look at input signal as sum of sine signals each with different frequence and phase and digital filter changes amplitude and phase of those components.
- We will demonstrate this by using following low-pass filter:

$$y(n) = x(n)+x(n-1) \tag{1.1}$$

where: y(n) – value of input signal at n-th sample,
x(n) – value of output signal at n-th sample,
n – n-th sample, n=1,2,3,...

- Eqaution (1.2) only works with samples and these are connected to real continuous signals with:

$$t=n\tau \tag{1.3}$$

where: t – time in seconds,
τ – sampling interval, time in seconds between two consecutive samples.

- In time domain, equation (1.1) looks like this:

$$y(t) = x(t)+x(t-\tau) \tag{1.4}$$

where: y(t) – value of input signal at time t,
x(t) – value of output signal at time t.

- We would also need following relations:

$$\omega = \frac{2\pi}{T} \tag{1.5}$$

$$f = \frac{1}{T} \tag{1.6}$$

$$f_s = \frac{1}{\tau} \tag{1.7}$$

where: T – signal period,
ω – signal frequency in radians per second,
f – signal frequency in periods per second,
f_s – sampling freq

1.2. Sine-wave analysis

– Sine-wave analysis is procedure of using sinusoid functions as input and observing how amplitude and phase are changed.

1.2.1. Experimental sine-wave analysis

– Experimental sine-wave analysis is procedure of using sinusoid function with some exact frequency, inserting it into filter equation like (1.4) and calculating filter output as sequence of numbers.
– These numbers are then drawn and amplitude and phase shift are calculated from the resulting picture.

- **Testing filter with single frequency**

– We can find out how our filter changes some frequency by using sine signal of that frequency as input and observing which signal is created at output as shown on figure 1.1.

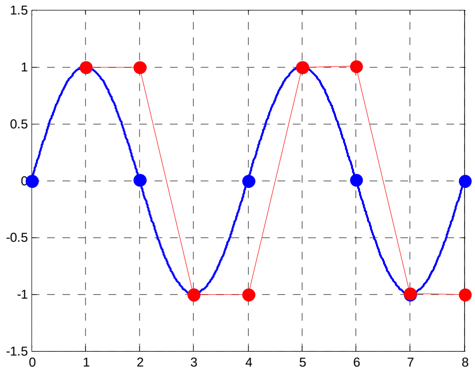


Figure 1.1. $f=f_s/4$

– Amplitude of filtered signal on figure 1.1. is calculated by observing that it reaches value 1 after 0.5s, from t=4.5s to t=5s:

$$f_s = 1 \text{ sample per second}$$
$$f = \frac{f_s}{4} = \frac{1}{4}$$
$$T = \frac{1}{f} = 4$$
$$A\sin(\omega t) = 1$$
$$A\sin(\frac{2\pi}{T}t) = A\sin(\frac{2\pi}{4}\frac{1}{2}) = A\sin(\frac{\pi}{4}) = A0.706 = 1$$
$$A = \frac{1}{0.706} = 1.414$$

– Filtered signal is shifted to the right by 0.5s since filtered signal crosses zero 0.5 seconds after original signal:

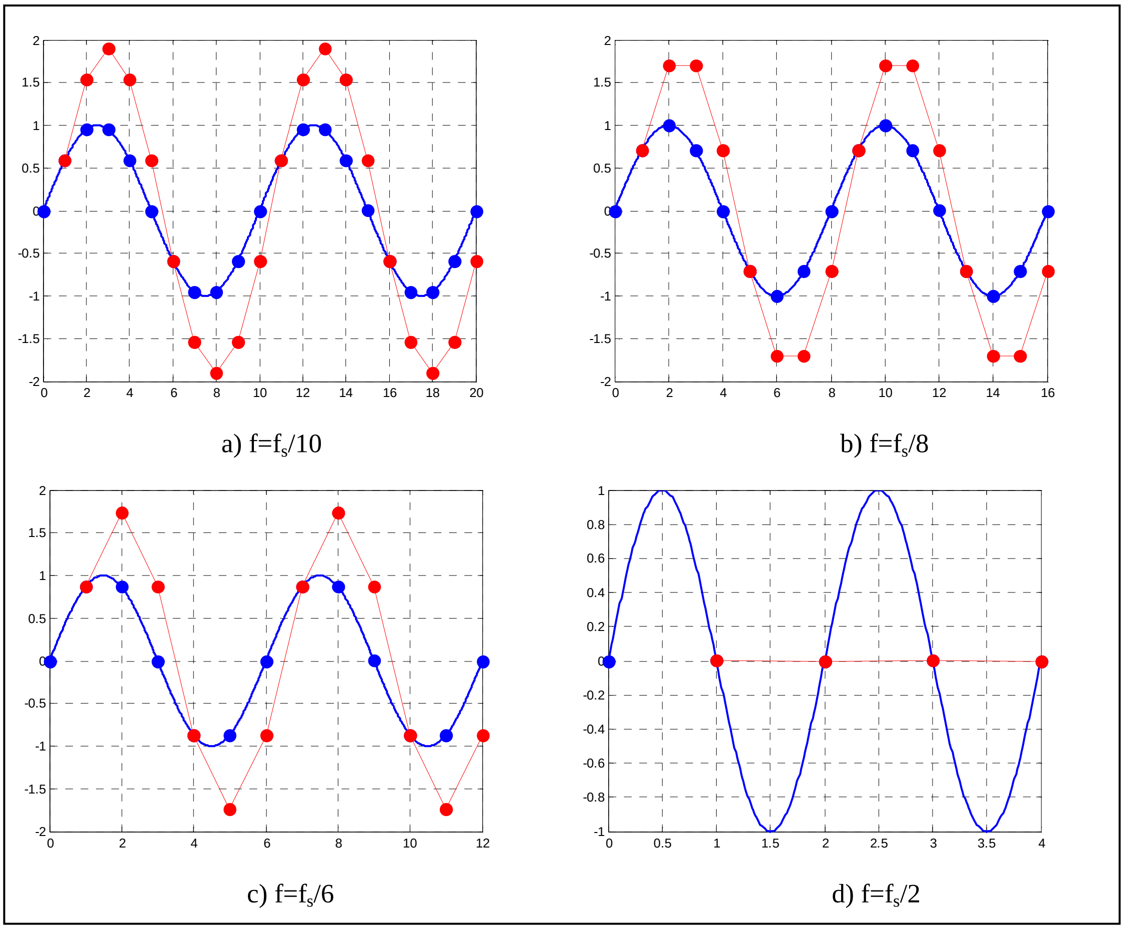
$$\varphi = \frac{2\pi}{T}t = \frac{2\pi}{4}0.5 = \frac{2\pi}{4}\frac{1}{2} = \frac{\pi}{4}\text{rad}$$

– This means that when original signal is at φ=0 filtered signal is is at -π/4 so we can finally write.

$$y(t) = 1.414\sin(\omega t - \frac{\pi}{4}) \quad , \quad f=f_s/4$$

- **Testing filter with multiple frequencies**

– Sine-wave analysis is procedure of testing filter at each frequency separately like we just did for $f=f_s/4$.



a) $f=f_s/10$ b) $f=f_s/8$ c) $f=f_s/6$ d) $f=f_s/2$

Figure 1.2. Sine-wave analysis.

– For figure 1.2.a) we have A=1.9 and since filtered signal is shifted to the right by 0.5s we also have:

$$f_s = 1 \text{ samples/s} \quad \Rightarrow \quad f = \frac{f_s}{10} = \frac{1}{10} \quad \Rightarrow \quad T = \frac{1}{f} = 10$$

$$\varphi = \frac{2\pi}{T} t = \frac{2\pi}{10}\frac{1}{2} = \frac{\pi}{10} \text{rad}$$

– For figure 1.2.b) we have:

$$f_s = 1 \text{ samples/s} \quad \Rightarrow \quad f = \frac{f_s}{8} = \frac{1}{8} \quad \Rightarrow \quad T = \frac{1}{f} = 8$$

$$A\sin(\frac{2\pi}{T}t) = A\sin(\frac{2\pi}{8}1.5) = A\sin(\frac{2\pi}{8}\frac{3}{2}) = A\sin(\frac{3\pi}{8}) = A0.9237 = 1.707$$

$$A = \frac{1.707}{0.9237} = 1.84$$

and since filtered signal is shifted to the right by 0.5s we also have:

$$f_s = 1 \text{ samples/s} \quad \Rightarrow \quad f = \frac{f_s}{8} = \frac{1}{8} \quad \Rightarrow \quad T = \frac{1}{f} = 8$$

$$\varphi = \frac{2\pi}{T} t = \frac{2\pi}{8} 0.5 = \frac{2\pi}{8}\frac{1}{2} = \frac{\pi}{8} \text{rad}$$

– For figure 1.2.c) we have A=1.73 and since filtered signal is shifted to the right by 0.5s we also have:

$$f_s = 1 \text{ samples/s} \quad \Rightarrow \quad f = \frac{f_s}{6} = \frac{1}{6} \quad \Rightarrow \quad T = \frac{1}{f} = 6$$

$$\varphi = \frac{2\pi}{T} t = \frac{2\pi}{6} 0.5 = \frac{2\pi}{6}\frac{1}{2} = \frac{\pi}{6} \text{rad}$$

– For figure 1.2.d) we have A=0 and φ=–π/2.

– All this results put together represent frequency response:

f	=	SF./	[10000	10	8	6	4	2];
A	=		[2	1.9	1.84	1.73	1.414	0];
φ	=	-pi./	[10000	10	8	6	4	2];

which can be graphicly presented like this:

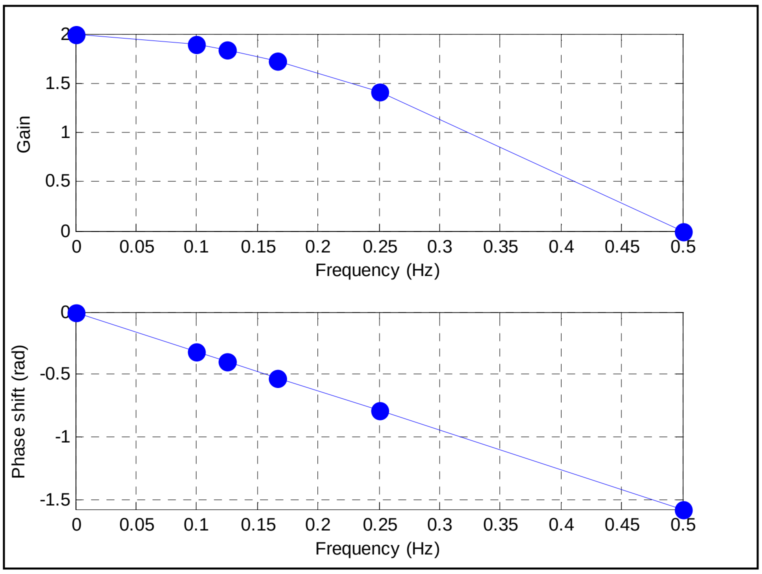


Figure 1.3. Frequency response using sine-wave analysis.

– Frequency response on figure 1.3. can be used to aproximate gain and phase shift of digital filter for any other frequency.

1.2.2. Mathematical sine-wave analysis using trigonometric functions

– We will now insert generic sine function in (1.4) in order to get mathemtical expresion for gain and phase shift.

- **Inserting generic sinusoid function**

– Generic sine function is defined like this:

$$x(t) = \sin(\omega t) \tag{1.8}$$

– Inserting (1.8) into (1.4) we get:

$$y(t) = \sin(\omega t)+ \sin[\omega(t-\tau)]$$

$$y(t) = \sin(\omega t)+ \sin(\omega t-\omega\tau) \tag{1.9}$$

– Using (2.8) from [2] we get:

$$y(t) = \sin(\omega t)+ \sin(\omega t)\cos(-\omega\tau)+\cos(\omega t)\sin(-\omega\tau)$$

$$y(t) = \sin(\omega t)+ \sin(\omega t)\cos(\omega\tau)-\cos(\omega t)\sin(\omega\tau)$$

$$y(t) = [1+\cos(\omega\tau)]\sin(\omega t)-\sin(\omega\tau)\cos(\omega t) \tag{1.10}$$

– Using following substitutions:

$$a = 1+\cos(\omega\tau) \tag{1.11}$$

$$b = -\sin(\omega\tau) \tag{1.12}$$

equation (1.10) can be written as:

$$y(t) =a\cdot\sin(\omega t)+b\cdot\cos(\omega t) \tag{1.13}$$

- **Define how should result look**

– We are looking for result of the form:

$$y(t) =G\sin(\omega t+\Theta) \tag{1.14}$$

where: $G(\omega)$ – gain which depends on frequency,
$\Theta(\omega)$ – phase shift which depends on frequency.

– Using (2.8) from [2], (1.14) can be written like:

$$y(t) =G[\sin(\omega t)\cos(\Theta)+\cos(\omega t)\sin(\Theta)]$$

$$y(t) =G\sin(\omega t)\cos(\Theta)+G\cos(\omega t)\sin(\Theta)$$

$$y(t) =G\cos(\Theta)\sin(\omega t) +G\sin(\Theta)\cos(\omega t) \tag{1.15}$$

– Using following substitutions:

$$a = G\cos(\Theta) \tag{1.16}$$

$$b = G\sin(\Theta) \tag{1.17}$$

equation (1.15) can be rewritten like:

$$y(t) =a\cdot\sin(\omega t) +b\cdot\cos(\omega t) \tag{1.18}$$

- **Compare results**

– Since (1.13) and (1.18) look the same that means that (1.11) equals (1.16) and that (1.12) equals (1.17).

- **Calculating Gain**

– Using (1.16) and (1.17) we get:

$$a^2+b^2 = G^2\cos^2(\Theta)+G^2\sin^2(\Theta) = G^2[\cos^2(\Theta)+\sin^2(\Theta)]$$

$$a^2+b^2 = G^2$$

– Inserting (1.11) and (1.12) we get:

$$G^2 = a^2+b^2 = [1+\cos(\omega\tau)]^2+\sin^2(\omega\tau) = 1+2\cos(\omega\tau)+\cos^2(\omega\tau)+\sin^2(\omega\tau) = 1+2\cos(\omega\tau)+1 = 2+2\cos(\omega\tau)$$

$$G^2= 2[1+\cos(\omega\tau)]$$

– Using (2.15) from [2] we get:

$$G^2 = 2\cdot 2\cos^2\left(\frac{\omega\tau}{2}\right) = 4\cos^2\left(\frac{\omega\tau}{2}\right)$$

$$G = 2\left|\cos\left(\frac{\omega\tau}{2}\right)\right|$$

– For $0\le f\le f_s/2$ we can write:

$$G = 2\cos\left(\frac{\omega\tau}{2}\right)$$

– By using (1.5) to (1.7) phase shift can be rewritten like this:

$$G(f) = 2\cos\left(\pi\frac{f}{f_s}\right) \quad , \quad 0\le f\le f_s/2 \qquad (1.19)$$

- **Calculating Phase shift**

– Using (1.16) and (1.17) we get:

$$\frac{b}{a} = \frac{G\sin(\Theta)}{G\cos(\Theta)} = \frac{\sin(\Theta)}{\cos(\Theta)}$$

$$\frac{b}{a} = \tan(\Theta)$$

– Inserting (1.11) and (1.12) we get:

$$\tan(\Theta) = \frac{b}{a} = \frac{-\sin(\omega\tau)}{1+\cos(\omega\tau)}$$

– Using (2.14) from [2] we can rewrite denumerator, and using (2.13) from [2] we can rewrite numerator like this:

$$\tan(\Theta) = -\frac{2\sin\left(\frac{\omega\tau}{2}\right)\cos\left(\frac{\omega\tau}{2}\right)}{1+\cos^2(\frac{\omega\tau}{2})-\sin^2(\frac{\omega\tau}{2})}$$

– Using (2.5) from [2] we get:

$$\tan(\Theta) = -\frac{2\sin\left(\frac{\omega\tau}{2}\right)\cos\left(\frac{\omega\tau}{2}\right)}{2\cos^2\left(\frac{\omega\tau}{2}\right)} = -\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2}\right)} = \frac{\sin\left(-\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2}\right)} = \tan\left(-\frac{\omega\tau}{2}\right)$$

– By using (1.5) to (1.7) phase shift can be rewritten like this:

$$\Theta(f) = -\pi\frac{f}{f_s} \qquad (1.20)$$

- **Display frequency response.**

– For defined low pass filter we have just calculated frequency response:

$$y(n) = x(n)+x(n-1) \tag{1.21}$$

$$G(\omega) = 2\cos(\pi f \tau) \quad , \quad 0 \le f \le f_s/2 \tag{1.22}$$

$$\Theta(\omega) = -\pi f \tau \tag{1.23}$$

– Frequency response defined by (1.22) and (1.23) looks like this:

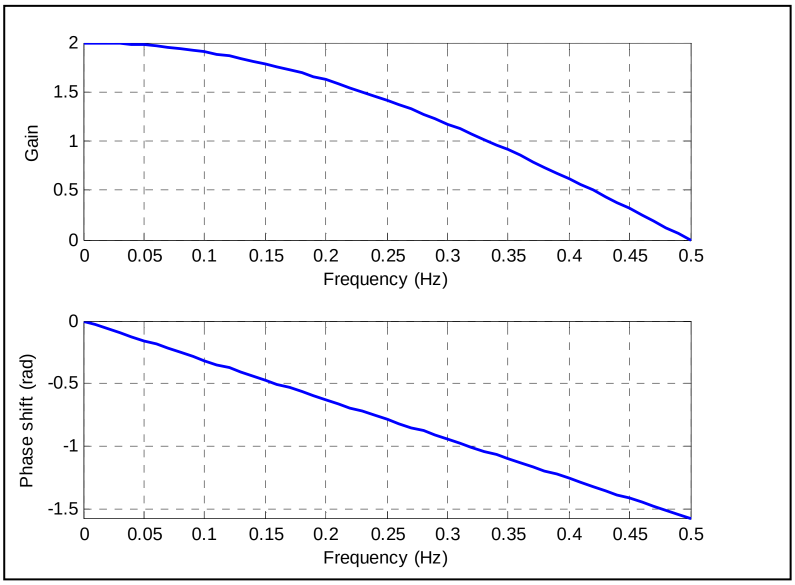


Figure 1.4. Frequency response.

1.2.3. Mathematical sine-wave analysis using complex sinusoid

– Complex sinusoid is defined like this:

$$x(t) = e^{j\omega t} \tag{1.24}$$

– Inserting (1.24) into (1.4) we get:

$$y(t) = e^{j\omega t} + e^{j\omega(t-\tau)} = e^{j\omega t} + e^{j\omega t - j\omega\tau} = e^{j\omega t} + e^{j\omega t}e^{-j\omega\tau}$$

$$y(t) = e^{j\omega t}\left(1 + e^{-j\omega\tau}\right)$$

– Inserting (1.24) we get:

$$y(t) = \left(1 + e^{-j\omega\tau}\right)x(t) \tag{1.25}$$

– First part of (1.25) is frequency response:

$$H(\omega) = 1 + e^{-j\omega\tau} \tag{1.26}$$

$$H(\omega) = G(\omega)e^{j\Theta(\omega)} \tag{1.27}$$

– Equation (1.26) can be transformed into (1.27) like this:

$$H(\omega) = \left(e^{j\omega\frac{\tau}{2}} + e^{-j\omega\frac{\tau}{2}}\right)e^{-j\omega\frac{\tau}{2}} \tag{1.28}$$

– Inserting (2.7) from [2] w get:

$$H(\omega) = 2\cos\left(\omega\frac{\tau}{2}\right)e^{-j\omega\frac{\tau}{2}} \tag{1.29}$$

– By applying (1.27) weg can calculate gain and phase shift:

$$G(\omega) = 2\cos\left(\omega\frac{\tau}{2}\right) \tag{1.30}$$

$$\Theta(\omega) = -\omega\frac{\tau}{2} \tag{1.31}$$

– By using (1.5) to (1.7) gain and phase shift can be rewritten like this:

$$G(f) = 2\cos\left(\pi\frac{f}{f_s}\right) \tag{1.32}$$

$$\Theta(f) = -\pi\frac{f}{f_s} \tag{1.33}$$

1.3. MATLAB scripts

1.3.1. Figure 1.1.

– This script shows how simple low pass filter works.

```
SF     = 1;      %Sampling frequency is 10 samples per second.

%CONTINUOUS SIGNAL.
f      = SF/4;         %Signal frequency is number of periods per second.
T      = 1/f;           %Signal period in seconds.
w      = 2*pi/T;        %Frequency in radians per second.
step   = 0.01;
t_cont = (0:step:2*T);  %Time in seconds. Show 2 periods of signal.
f_cont = sin(w*t_cont);
figure(1); hold off; plot(t_cont, f_cont,'LineWidth',2); grid;

%DESCRETE SIGNAL.
SI     = 1/SF;    %Sampling interval is number of seconds between two samples.
t_desc = t_cont(1:SI/step:length(t_cont));  %Take samples at this times.
f_desc = f_cont(1:SI/step:length(f_cont)); %Descrete signal containing only samples of continuous signal.
figure(1); hold on; plot(t_desc, f_desc,'.','MarkerSize',30);

%FILTERED DESCRETE SIGNAL.
f_filtered = f_desc(2:length(f_desc))+f_desc(1:length(f_desc)-1);
figure(1); hold on; plot(SI+t_desc(1:length(t_desc)-1), f_filtered,'-r.','MarkerSize',30);
```

1.3.2. Figure 1.3.

– This script shows frequency response obtained using sine-wave method.

```
%FREQUENCY RESPONSE.
figure(2); hold off;
SF = 1;
f  = SF./ [10000 10  8    6     4     2];
A  =      [2     1.9 1.84 1.73  1.414 0];
fi = -pi./[10000 10  8    6     4     2];
subplot(2,1,1); plot(f, A,'-b.','MarkerSize',30);  grid on; xlabel('Frequency (Hz)'); ylabel('Gain');
axis([0 0.5 0 2]);
subplot(2,1,2); plot(f, fi,'-b.','MarkerSize',30); grid on; xlabel('Frequency (Hz)'); ylabel('Phase shift
(rad)'); axis([0 0.5 -pi/2 0]);
```

1.3.3. Figure 1.4.

– This script shows frequency response obtained using mathematical sine-wave method.

```
fs    = 1;
tau   = 1/fs;
f     = (0:0.01:fs/2);
Gain  = 2*cos(pi*f*tau);
Phase = -pi*f*tau;
figure(1); hold off;
subplot(2,1,1); plot(f,Gain ,'LineWidth',1); grid on; xlabel('Frequency (Hz)'); ylabel('Gain');
subplot(2,1,2); plot(f,Phase,'LineWidth',1); grid on; xlabel('Frequency (Hz)'); ylabel('Phase shift (rad)'); axis([0 0.5 -pi/2 0]);
```

1.4. References

[1] http://ccrma.stanford.edu/~jos/filters

[2] Trigonometric functions